

# Popular Distributions

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## Notation

- notation

RV distributed as  $\text{dist\_name}(\text{parameters})$

$\sim$

pdf or pf:

$$f(x | \text{parameters}) = c \times k(x)$$

normalization constant

$$\int_X k(x) dx = \frac{1}{c}$$

# Moments

$$\text{mean} = \mathcal{E}(X) = \int_X x f_X(x) dx$$

$$\text{var}(X) = \int_X (x - \mathcal{E}(X))^2 f_X(x) dx = \int_X x^2 f_X(x) dx - (\mathcal{E}(X))^2$$

$$\text{cov}(X_i, X_j) = \int_{X_i} \int_{X_j} (x_i - \mathcal{E}(X_i))(x_j - \mathcal{E}(X_j)) f(x_i, x_j) dx_i dx_j$$

$$= \int_{X_1} \int_{X_2} \cdots \int_{X_p} (x_i - \mathcal{E}(X_i))(x_j - \mathcal{E}(X_j)) f_X(x_1, x_2, \dots, x_p) dx_p \cdots dx_2 dx_1$$

$$\mathcal{D}(\mathbf{X}) = \begin{bmatrix} \text{var}(X_1) & \text{cov}(X_1, X_2) & \cdots & \text{cov}(X_1, X_j) & \cdots & \text{cov}(X_1, X_p) \\ \text{cov}(X_2, X_1) & \text{var}(X_2) & \cdots & \text{cov}(X_2, X_j) & \cdots & \text{cov}(X_2, X_p) \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ \text{cov}(X_i, X_1) & \text{cov}(X_i, X_2) & \cdots & \text{cov}(X_i, X_j) & \cdots & \text{cov}(X_i, X_p) \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ \text{cov}(X_p, X_1) & \text{cov}(X_p, X_2) & \cdots & \text{cov}(X_p, X_j) & \cdots & \text{var}(X_p) \end{bmatrix}$$

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# Moments

- calculation of the dispersion matrix

$$\mathcal{E}(\mathbf{X}) = (\mathcal{E}(X_1), \mathcal{E}(X_2), \dots, \mathcal{E}(X_p))'$$

$$\mathcal{D}(\mathbf{X}) = \mathcal{E}(\mathbf{X}\mathbf{X}') - \mathcal{E}(\mathbf{X})\mathcal{E}(\mathbf{X})'$$

- using conditional expectation and dispersion

$$\mathcal{E}(X_1) = \mathcal{E}(\mathcal{E}(X_1|X_2))$$

$$\text{var}(X_1) = \mathcal{E}(\text{var}(X_1|X_2)) + \text{var}(\mathcal{E}(X_1|X_2))$$

$$\mathcal{E}(\mathbf{X}_1) = \mathcal{E}(\mathcal{E}(\mathbf{X}_1|\mathbf{X}_2))$$

$$\mathcal{D}(\mathbf{X}_1) = \mathcal{E}(\mathcal{D}(\mathbf{X}_1|\mathbf{X}_2)) + \mathcal{D}(\mathcal{E}(\mathbf{X}_1|\mathbf{X}_2))$$

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# Posterior Variance

$$\text{var}(X_1) = \mathcal{E}(\text{var}(X_1|X_2)) + \text{var}(\mathcal{E}(X_1|X_2))$$

$$X_1 \rightarrow \theta \quad X_2 \rightarrow X$$

$$\mathcal{E}_X[\text{var}(\theta|X)] = \text{var}(\theta) - \text{var}[\mathcal{E}(\theta|X)]$$

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## Mode of pdf

- mode = value of RV at which pdf is maximum.

$$\frac{d f_X(x)}{d x} = 0 \quad \text{or} \quad \frac{d \log f_X(x)}{d x} = 0$$

$$\frac{\partial f(x, y)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f(x, y)}{\partial y} = 0$$

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# Bernoulli

- notation  $X \sim \text{Bernoulli}(\pi)$
- pf

$$f(x|\pi) = \pi^x (1 - \pi)^{1-x}$$

- moments

$$\mathcal{E}(X) = \pi \qquad \text{var}(X) = \pi(1 - \pi)$$

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# Binomial

- notation  $X \sim \text{Bin}(n, \pi)$
- pf

$$f(x|n, \pi) = {}_n C_x \times \pi^x (1 - \pi)^{n-x} = \frac{n!}{x!(n-x)!} \times \pi^x (1 - \pi)^{n-x}$$

- moments

$$\begin{aligned} \mathcal{E}(X) &= n\pi & \text{var}(X) &= n\pi(1 - \pi) \\ \text{mode}(X) &= [(n + 1)\pi] \end{aligned}$$

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# Multinomial

■ notation  $X \sim \text{Multin}(n, \boldsymbol{\pi})$

■ pf  $\sum_{j=1}^k x_j = n \quad \sum_{i=1}^k \pi_i = 1$

$$f(\mathbf{x}|n, \boldsymbol{\pi}) = \frac{n!}{x_1!x_2!\dots x_k!} \pi_1^{x_1} \pi_2^{x_2} \dots \pi_k^{x_k} = \frac{n!}{\prod_{i=1}^k x_i!} \prod_{i=1}^k \pi_i^{x_i}$$

■ moments

$$\mathcal{E}(X_i) = n\pi_i$$

$$\text{var}(X_i) = n\pi_i(1 - \pi_i)$$

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# Poisson

■ notation  $X \sim \text{Poisson}(\mu)$

■ pf

$$f(x|\mu) = \exp(-\mu) \times \frac{\mu^x}{x!}$$

■ moments

$$\mathcal{E}(X) = \mu \quad \text{var}(X) = \mu$$

$$\text{mode} = \mu - 1 \quad \text{if } \mu > 1$$

■ comment

$$f(x|\mu) = \frac{n!}{x!(n-x)!} \left(\frac{\mu}{n}\right)^x \left(1 - \frac{\mu}{n}\right)^{n-x}$$

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# Normal

- notation  $X \sim N(\mu, \sigma^2)$
- pdf

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \times \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$$

- moments

$$\mathcal{E}(X) = \mu \quad \text{var}(X) = \sigma^2$$

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# Multivariate Normal

- notation  $X \sim MVN(\mu, \Sigma)$
- pdf

$$f(\mathbf{x}|\mu, \Sigma) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \times \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\right)$$

- example

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \quad \rho = \frac{\sigma_{12}}{\sigma_1\sigma_2}$$

$$\frac{1}{2\pi\sqrt{1-\rho^2}} \times \exp\left(-\frac{1}{2(1-\rho^2)} \left[ \left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 \right]\right)$$

- moments

$$\mathcal{E}(X) = \mu \quad \mathcal{D}(X) = \Sigma$$

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# Multivariate Normal (cont.)

- linear combinations

$$AX + b \sim N(b + A\mu, A\Sigma A')$$

- from conditional and marginal

$$y_2|y_1 \sim MVN(\gamma_2 + Gy_1, \Psi)$$

$$y_1 \sim MVN(\gamma_1, \Phi)$$

$$y_2 \sim MVN(\gamma_2 + G\gamma_1, G\Phi G' + \Psi)$$

- distribution of the quadratic form

$$Q = \frac{1}{p}(x - \mu)' \Sigma^{-1}(x - \mu) \sim \chi^2(p, 1)$$

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## t

- notation  $X \sim t(\nu, \gamma, c)$

- pdf

$$f(x|\nu, \gamma, c) = \frac{\Gamma(\frac{1}{2}(\nu + 1))}{\Gamma(\frac{1}{2})\Gamma(\frac{\nu}{2})} \nu^{-\frac{1}{2}} c^{-\frac{1}{2}} \times \left(1 + \frac{1}{\nu c}(x - \gamma)^2\right)^{-\frac{1}{2}(\nu + 1)}$$

- moments

$$\mathcal{E}(X) = \gamma \quad \text{if } \nu > 1$$

$$\text{var}(X) = \frac{\nu}{\nu - 2} c \quad \text{if } \nu > 2$$

- distribution of the quadratic form

$$Q = \frac{1}{c}(X - \gamma)^2 \sim F(1, \nu)$$

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# Multivariate t

- notation  $X \sim MVt(\nu, \gamma, C)$

- pdf

$$f(x|\nu, \gamma, C) = \frac{\Gamma(\frac{1}{2}(\nu + p))}{[\Gamma(\frac{1}{2})]^p \Gamma(\frac{\nu}{2})} \nu^{-\frac{1}{2}p} |C|^{-\frac{1}{2}} \times \left(1 + \frac{1}{\nu} (x - \gamma)' C^{-1} (x - \gamma)\right)^{-\frac{1}{2}(\nu + p)}$$

- moments

$$\mathcal{E}(X) = \mu, \quad \mathcal{D}(X) = \frac{\nu}{\nu - 2} C$$

- distribution of the quadratic form

$$Q = \frac{1}{p} (x - \gamma)' C^{-1} (x - \gamma) \sim F(p, \nu)$$

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# Gamma

- notation  $X \sim \Gamma(\alpha, \beta)$

- pdf

$$f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \times x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right)$$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \exp(-x) dx$$

- moments

$$\mathcal{E}(X) = \alpha\beta \quad \text{var}(X) = \alpha\beta^2$$

$$\text{mode}(X) = \beta(\alpha - 1)$$

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# Chi Squared

- notation  $X \sim \Gamma\left(\frac{\nu}{2}, 2\right) \longleftrightarrow X \sim \chi^2(\nu)$
- pdf

$$f(x|\nu) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right)2^{\frac{\nu}{2}}} \times x^{\frac{\nu}{2}-1} \exp\left(-\frac{x}{2}\right)$$

- moments

$$\begin{aligned}\mathcal{E}(X) &= \nu\kappa^2 & \text{var}(X) &= 2\nu(\kappa^2)^2 \\ \text{mode}(X) &= \kappa^2(\nu - 2)\end{aligned}$$

- relationship

$$X/\kappa^2 \sim \chi^2(\nu), \quad \frac{1}{X} \sim \chi^{-2}\left(\nu, \frac{1}{\kappa^2}\right), \quad \sqrt{X} \sim \chi(\nu, \kappa^2)$$

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# Inverted Chi Squared

- notation  $X \sim \chi^{-2}(\nu, \kappa^2) \quad \kappa^2/X \sim \chi^2(\nu)$
- pdf

$$f(x|\nu, \kappa^2) = \frac{(\kappa^2)^{\frac{\nu}{2}}}{2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)} \times x^{-\frac{\nu}{2}-1} \exp\left(-\frac{\kappa^2}{2x}\right)$$

- moments

$$\begin{aligned}\mathcal{E}(X) &= \frac{\kappa^2}{\nu - 2} & \text{var}(X) &= \frac{2(\kappa^2)^2}{(\nu - 2)^2(\nu - 4)} \\ \text{mode}(X) &= \frac{\kappa^2}{\nu + 2}\end{aligned}$$

- relationship

$$\frac{\kappa^2}{X} \sim \chi^2(\nu), \quad \frac{1}{X} \sim \chi^{-2}\left(\nu, \frac{1}{\kappa^2}\right), \quad \sqrt{X} \sim \chi^{-1}(\nu, \kappa^2)$$

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# Inverted Chi

- notation  $X \sim \chi^{-1}(\nu, \kappa^2)$       $\kappa^2/X^2 \sim \chi^2(\nu)$
- pdf

$$f(x|\nu, \kappa^2) = \frac{(\kappa^2)^{\frac{\nu}{2}}}{2^{\frac{\nu}{2}-1}\Gamma(\frac{\nu}{2})} \times x^{-\nu-1} \exp\left(-\frac{\kappa^2}{2x^2}\right)$$

- moments

$$\begin{aligned} \mathcal{E}(X) &= \frac{\kappa\Gamma(\frac{\nu-1}{2})}{\sqrt{2}\Gamma(\frac{\nu}{2})} \approx \frac{\kappa}{\sqrt{\nu - \frac{3}{2}}} & \text{var}(X) &= \frac{\kappa^2}{\nu-2} - (\mathcal{E}(X))^2 \\ \text{mode}(X) &= \frac{\kappa}{\sqrt{\nu+1}} & &\approx \frac{\kappa^2}{2(\nu-2)(\nu - \frac{5}{3})} \end{aligned}$$

- relationship

$$\frac{\kappa^2}{X^2} \sim \chi^2(\nu), \quad X^2 \sim \chi^{-2}(\nu, \kappa^2), \quad \frac{1}{X} \sim \chi\left(\nu, \frac{1}{\kappa^2}\right)$$

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# F

- notation  $X \sim F(\nu_1, \nu_2)$       $\frac{1}{X} \sim F(\nu_2, \nu_1)$
- pdf

$$f(x|\nu_1, \nu_2) \sim \frac{\nu_1^{\nu_1/2} \nu_2^{\nu_2/2}}{B(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2)} \frac{x^{\frac{\nu_1}{2}-1}}{(\nu_2 + \nu_1 x)^{(\nu_1+\nu_2)/2}}$$

- moments

$$\begin{aligned} \mathcal{E}(X) &= \frac{\nu_2}{\nu_2 - 2}, & \text{var}(X) &= \frac{2\nu_2^2(\nu_1 + \nu_2 - 2)}{\nu_1(\nu_2 - 2)^2(\nu_2 - 4)} \\ \text{mode}(X) &= \frac{\nu_2}{\nu_2 + 1} \frac{\nu_1 - 2}{\nu_1} \end{aligned}$$

- relationship

$$\begin{aligned} x_1 &\sim \chi^{-2}(\nu_1, \kappa_1^2) \\ x_2 &\sim \chi^{-2}(\nu_2, \kappa_2^2) \end{aligned} \quad F = \frac{\frac{x_2}{\kappa_2^2} \times \frac{1}{\nu_2}}{\frac{x_1}{\kappa_1^2} \times \frac{1}{\nu_1}}$$

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# Normal-Inverted Chi

- notation  $\begin{pmatrix} \mu \\ \sigma \end{pmatrix} \sim N\chi^{-1}(\gamma, \phi, \nu, \kappa^2)$
- pdf

$$f(\mu, \sigma | \gamma, \phi, \nu, \kappa^2) = h(\mu | \sigma)h(\sigma)$$

$$\propto \frac{1}{\sqrt{\sigma^2 \phi}} \exp\left(-\frac{1}{2\sigma^2}(\mu - \gamma)^2 \frac{1}{\phi}\right) \times \sigma^{-\nu-1} \exp\left(-\frac{\kappa^2}{2\sigma^2}\right) (\kappa^2)^{\frac{1}{\nu}}$$

$$\propto \sigma^{-\nu-2} \exp\left[-\frac{1}{2\sigma^2} \left(\frac{1}{\phi}(\mu - \gamma)^2 + \kappa^2\right)\right]$$

- definition

$$\mu | \sigma \sim N(\gamma, \sigma^2 \phi)$$

$$\sigma \sim \chi^{-1}(\nu, \kappa^2)$$

$$\kappa^2 / \sigma^2 \sim \chi^2(\nu)$$

$$\mu \sim t(\nu, \gamma, \frac{\kappa^2 \phi}{\nu})$$

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# Normal-Inverted Chi (cont.)

- marginal

$$h(\mu) = \int_0^\infty h(\mu, \sigma) d\sigma \propto \int_0^\infty \sigma^{-\nu-2} \exp\left[-\frac{1}{2\sigma^2} \left(\frac{1}{\phi}(\mu - \gamma)^2 + \kappa^2\right)\right] d\sigma$$

$$= \frac{2^{\frac{\nu-1}{2}} \Gamma(\frac{\nu+1}{2})}{\left(\frac{1}{\phi}(\mu - \gamma)^2 + \kappa^2\right)^{\frac{\nu+1}{2}}}$$

$$\propto \frac{1}{\left(\frac{1}{\phi}(\mu - \gamma)^2 + \kappa^2\right)^{\frac{\nu+1}{2}}}$$

$$\propto \left(\frac{1}{\kappa^2 \phi}(\mu - \gamma)^2 + 1\right)^{-\frac{1}{2}(\nu+1)}$$

$$\mu \sim t(\nu, \gamma, \frac{\kappa^2 \phi}{\nu})$$

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# Normal-Inverted Chi (cont.)

- moments

$$\begin{aligned}
 \mathcal{E}(\mu) &= \gamma & \text{var}(\mu) &= \frac{\kappa^2 \phi}{\nu - 2} \\
 \text{joint mode}(\mu) &= \text{marginal mode}(\mu) = \gamma \\
 \mathcal{E}(\sigma) &\approx \frac{\kappa}{\nu - \frac{3}{2}} & \text{var}(\sigma) &\approx \frac{\kappa^2}{2(\nu - 2)(\nu - \frac{5}{3})} \\
 \text{joint mode}(\sigma) &= \frac{\kappa}{\sqrt{\nu + 2}} & \text{marginal mode}(\sigma) &= \frac{\kappa}{\sqrt{\nu + 1}}
 \end{aligned}$$

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# Multivariate Normal-Inverted Chi

- notation  $\begin{pmatrix} \beta \\ \sigma \end{pmatrix} \sim MVN\chi^{-1}(\gamma, \Phi, \nu, \kappa^2)$

- pdf

$$\begin{aligned}
 h(\beta, \sigma) &\propto \frac{1}{\sigma^p \sqrt{|\Phi|}} \exp\left(-\frac{1}{2\sigma^2}(\beta - \gamma)' \Phi^{-1}(\beta - \gamma)\right) \times \\
 &\quad \frac{1}{2^{\frac{\nu}{2}-1} \Gamma(\frac{\nu}{2})} \exp\left(-\frac{\kappa^2}{2\sigma^2}\right) (\kappa^2)^{\frac{1}{\nu}} \sigma^{-\nu-1} \\
 &= \frac{1}{2^{\frac{\nu}{2}-1} \Gamma(\frac{\nu}{2})} \frac{1}{\sqrt{|\Phi|}} \exp\left[-\frac{1}{2\sigma^2} \left((\beta - \gamma)' \Phi^{-1}(\beta - \gamma) + \kappa^2\right)\right] (\kappa^2)^{\frac{1}{\nu}} \sigma^{-\nu-p-1} \\
 &\propto \exp\left[-\frac{1}{2\sigma^2} \left((\beta - \gamma)' \Phi^{-1}(\beta - \gamma) + \kappa^2\right)\right] \sigma^{-\nu-p-1}
 \end{aligned}$$

- definition

$$\beta | \sigma \sim MVN(\gamma, \sigma^2 \Phi)$$

$$\sigma \sim \chi^{-1}(\nu, \kappa^2)$$

$$\kappa^2 / \sigma^2 \sim \chi^2(\nu)$$

$$\beta \sim t(\nu, \gamma, \frac{\kappa^2}{\nu} \Phi)$$

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## Multivariate Normal-Inverted Chi (cont.)

- marginal

$$\begin{aligned}h(\boldsymbol{\beta}) &= \int_0^\infty h(\boldsymbol{\beta}, \sigma) d\sigma \\ &\propto \int_0^\infty \exp\left[-\frac{1}{2\sigma^2} \left((\boldsymbol{\beta} - \boldsymbol{\gamma})' \boldsymbol{\Phi}^{-1} (\boldsymbol{\beta} - \boldsymbol{\gamma}) + \kappa^2\right)\right] \sigma^{-\nu-p-1} d\sigma \\ &\propto \left[(\boldsymbol{\beta} - \boldsymbol{\gamma})' \boldsymbol{\Phi}^{-1} (\boldsymbol{\beta} - \boldsymbol{\gamma}) + \kappa^2\right]^{-\frac{1}{2}(\nu+p)} \\ &\propto \left[\frac{1}{\nu} (\boldsymbol{\beta} - \boldsymbol{\gamma})' \left(\frac{\kappa^2}{\nu} \boldsymbol{\Phi}\right)^{-1} (\boldsymbol{\beta} - \boldsymbol{\gamma}) + 1\right]^{-\frac{1}{2}(\nu+p)}\end{aligned}$$

$$\boldsymbol{\beta} \sim t(\nu, \boldsymbol{\gamma}, \frac{\kappa^2}{\nu} \boldsymbol{\Phi})$$

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## Beta

- notation  $X \sim \mathbf{Beta}(\alpha, \beta)$

- pdf  $0 < x < 1$

$$f(x|\alpha, \beta) = \frac{1}{B(\alpha, \beta)} \times x^{\alpha-1} (1-x)^{\beta-1}$$

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

- moments

$$\mathcal{E}(X) = \frac{\alpha}{\alpha + \beta}, \quad \text{var}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2 (\alpha + \beta + 1)}$$

$$\text{mode}(X) = \frac{\alpha - 1}{\alpha + \beta - 2}$$

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# Dirichlet

- notation  $X \sim \mathbf{Dir}(\boldsymbol{\alpha})$

- pdf  $0 < X_i < 1, i = 1, 2, \dots, p$   $\sum_{i=1}^p X_i < 1$

$$f(\mathbf{x}|\boldsymbol{\alpha}) = \frac{\Gamma(\sum_{j=1}^{p+1} \alpha_j)}{\prod_{j=1}^{p+1} \Gamma(\alpha_j)} \times \left( \prod_{j=1}^p x_j^{\alpha_j-1} \right) \left( 1 - \sum_{j=1}^p x_j \right)^{\alpha_{p+1}-1}$$

- moments

$$\mathcal{E}(X_i) = \frac{\alpha_i}{\alpha_0}, \quad \text{var}(X_i) = \frac{\alpha_i(\alpha_0 - \alpha_i)}{\alpha_0^2(\alpha_0 + 1)}$$

$$\text{mode}(X_i) = \frac{\alpha_i - 1}{\alpha_0 - 2}$$

$$\text{cov}(X_i, X_j) = -\frac{\alpha_i \alpha_j}{\alpha_0^2(\alpha_0 + 1)}$$

$$\text{jointmode}(\mathbf{X}) = \left\{ \frac{\alpha_i - 1}{\alpha_0 - p - 1} \right\}$$

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## Dirichlet (cont.)

- marginal is beta.

$$X_i \sim B(\alpha_i, \alpha_0), \quad i = 1, 2, \dots, p + 1$$

$$\alpha_0 = \sum_{j=1}^{p+1} \alpha_j$$

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# Wishart

- notation  $\mathbf{X} \sim W(\nu, \Sigma)$   $\mathbf{X}$  is  $p \times p$  positive definite matrix.
- pdf

$$f(\mathbf{X}|\nu, \Sigma) = \frac{1}{2^{\frac{1}{2}(\nu p)} \pi^{\frac{1}{4}p(p-1)} |\Sigma|^{\frac{\nu}{2}} \prod_{i=1}^p \Gamma(\frac{1}{2}(\nu + 1 - i))} \times \\ |\mathbf{X}|^{\frac{1}{2}(\nu-p-1)} \exp\left(-\frac{1}{2}\text{tr}\left(\Sigma^{-1}\mathbf{X}\right)\right)$$

- moment

$$\mathcal{E}(\mathbf{X}) = \nu \Sigma$$

$$\text{var}(x_{ij}) = \nu(\sigma_{ij}^2 + \sigma_{ii}\sigma_{jj})$$

$$\text{mode}(\mathbf{X}) = (\nu - p - 1)\Sigma$$

$$\text{cov}(x_{ij}, x_{kl}) = \nu(\sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk})$$

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# Inverted Wishart

- notation  $\mathbf{X} \sim W^{-1}(\nu, \Sigma)$   $\mathbf{X}^{-1} \sim W(\nu, \Sigma^{-1})$
- pdf  $\mathbf{X}$  is  $p \times p$  positive definite matrix.

$$f(\mathbf{X}|\nu, \Sigma) = \frac{|\Sigma|^{\frac{1}{2}\nu}}{2^{\frac{1}{2}(\nu p)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^p \Gamma(\frac{1}{2}(\nu + 1 - i))} \times \\ \frac{1}{|\mathbf{X}|^{\frac{1}{2}(\nu+p+1)}} \exp\left(-\frac{1}{2}\text{tr}\left(\Sigma\mathbf{X}^{-1}\right)\right)$$

- moment

$$\mathcal{E}(\mathbf{X}) = \frac{1}{\nu - p - 1} \Sigma$$

$$\text{var}(x_{ii}) = \frac{2(\sigma_{ii})^2}{(\nu - p - 1)(\nu - p - 3)}$$

$$\text{mode}(X) = \frac{1}{\nu + p + 1} \Sigma$$

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