

vec of a Partitioned Matrix

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Let \mathbf{A} and \mathbf{B} be the usual matrix of the sizes, $(ra \times cb)$, and $(rb \times cb)$, resp.

Assume that $ra = rb$.

Horizontally augmented partitioned matrix of size $(ra \times (ca + cb))$.

$$\mathbf{C} = (\mathbf{A}, \mathbf{B}) \tag{1}$$

The vec of \mathbf{C} is

$$\text{vec}(\mathbf{C}) = \begin{pmatrix} \text{vec}(\mathbf{A}) \\ \text{vec}(\mathbf{B}) \end{pmatrix} \tag{2}$$

Now, assuming $ca = cb$, consider the vertically augmented partitioned matrix of size $((ra+rb) \times ca)$.

$$\mathbf{D} = \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \tag{3}$$

Note that

$$\text{vec}(\mathbf{D}) \neq \begin{pmatrix} \text{vec}(\mathbf{A}) \\ \text{vec}(\mathbf{B}) \end{pmatrix} \tag{4}$$

The vec of \mathbf{D} can be expressed as follows using the commutation matrix. The commutation matrix $K_{ca,ra}$ is defined as the matrix with which the relationship

$$\text{vec}(\mathbf{A}) = K_{ca,ra} \text{vec}(\mathbf{A}') \tag{5}$$

is true.

Using this, we have

$$\text{vec}(\mathbf{D}) = K_{cd,rd} \text{vec}(\mathbf{D}') \tag{6}$$

$$= K_{cd,rd} \text{vec}(\mathbf{A}', \mathbf{B}') \tag{7}$$

$$= K_{cd,rd} \begin{pmatrix} \text{vec}(\mathbf{A}') \\ \text{vec}(\mathbf{B}') \end{pmatrix} \tag{8}$$

Now partition the rows of $K_{ca,ra}$ into two blocks as

$$K_{ca,ra} = \begin{bmatrix} K1 \\ K2 \end{bmatrix} \tag{9}$$

Then, we have

$$\text{vec}(\mathbf{D}) = K_{cd,rd} \text{vec}(\mathbf{D}') \tag{10}$$

$$= \begin{bmatrix} K1 \\ K2 \end{bmatrix} \begin{pmatrix} \text{vec}(\mathbf{A}') \\ \text{vec}(\mathbf{B}') \end{pmatrix} \tag{11}$$

$$= \begin{pmatrix} K1 \begin{pmatrix} \text{vec}(\mathbf{A}') \\ \text{vec}(\mathbf{B}') \end{pmatrix} \\ K2 \begin{pmatrix} \text{vec}(\mathbf{A}') \\ \text{vec}(\mathbf{B}') \end{pmatrix} \end{pmatrix} \tag{12}$$

This is what we get as we use $\text{vecK}[D]$.

Now partition the columns of $K_{ca,ra}$ into two blocks as

$$K_{ca,ra} = \begin{bmatrix} K3 & K4 \end{bmatrix} \quad (13)$$

Then, we have

$$\text{vec}(\mathbf{D}) = K_{cd,rd}\text{vec}(\mathbf{D}') \quad (14)$$

$$= \begin{bmatrix} K3 & K4 \end{bmatrix} \begin{pmatrix} \text{vec}(\mathbf{A}') \\ \text{vec}(\mathbf{B}') \end{pmatrix} \quad (15)$$

$$= (K3\text{vec}(\mathbf{A}') + K4\text{vec}(\mathbf{B}')) \quad (16)$$

This may be more compact.