

## 付録 いくつかの結果

以下に、無制約な  $X$  行列が 2 個まで含まれる場合の例を示す。トレースの演算規則を用いて生成される同値な結果のうち、 $X$  や  $X'$  の出現位置が質的に異なるもののみを列挙した。ここに示した例よりも単純なケースは、 $A, B, C$  等を単位行列とおくことによって得られる。なお、 $X$  が対称行列の場合には、以下の結果に (32) 式を適用すればよい。

$$\begin{aligned} \frac{\partial \operatorname{tr}(AXB)}{\partial X} &= \frac{\partial \operatorname{tr}(BAX)}{\partial X} = \frac{\partial \operatorname{tr}(XBA)}{\partial X} = \frac{\partial \operatorname{tr}(B'X'A')}{\partial X} \\ &= \frac{\partial \operatorname{tr}(A'B'X')}{\partial X} = \frac{\partial \operatorname{tr}(X'A'B')}{\partial X} = A'B' \end{aligned} \quad (140)$$

$$\begin{aligned} \frac{\partial \operatorname{tr}(AX' BXC)}{\partial X} &= \frac{\partial \operatorname{tr}(CAX' BX)}{\partial X} = \frac{\partial \operatorname{tr}(XCAX' B)}{\partial X} \\ &= \frac{\partial \operatorname{tr}(BXCAX')}{\partial X} = \frac{\partial \operatorname{tr}(X' BXC A)}{\partial X} \\ &= BXCA + B'X A'C' \end{aligned} \quad (141)$$

$$\begin{aligned} \frac{\partial \operatorname{tr}(AXBXC)}{\partial X} &= \frac{\partial \operatorname{tr}(CAXBX)}{\partial X} = \frac{\partial \operatorname{tr}(XCAXB)}{\partial X} \\ &= \frac{\partial \operatorname{tr}(BXCAX)}{\partial X} = \frac{\partial \operatorname{tr}(C'X'B'X'A')}{\partial X} \\ &= \frac{\partial \operatorname{tr}(A'C'X'B'X')}{\partial X} = \frac{\partial \operatorname{tr}(X'A'C'X'B')}{\partial X} \\ &= (BXCA)' + (CAXB)' \end{aligned} \quad (142)$$

$$\begin{aligned} \frac{\partial \operatorname{tr}(A(BXC)^{-1}D)}{\partial X} &= \frac{\partial \operatorname{tr}(DA(BXC)^{-1})}{\partial X} = \frac{\partial \operatorname{tr}((BXC)^{-1}DA)}{\partial X} \\ &= \frac{\partial \operatorname{tr}(D'(C'X'B')^{-1}A')}{\partial X} = \frac{\partial \operatorname{tr}(A'D'(C'X'B')^{-1})}{\partial X} \\ &= \frac{\partial \operatorname{tr}((C'X'B')^{-1}A'D')}{\partial X} \\ &= -\left(C(BXC)^{-1}DA(BXC)^{-1}B\right)' \end{aligned} \quad (143)$$

$$\begin{aligned} \frac{\partial \operatorname{tr}(A(X' BX)^{-1}C)}{\partial X} &= \frac{\partial \operatorname{tr}(CA(X' BX)^{-1})}{\partial X} = \frac{\partial \operatorname{tr}((X' BX)^{-1}CA)}{\partial X} \\ &= -BX(X' BX)^{-1}CA(X' BX)^{-1} \\ &\quad -B'X(X' BX)^{-1}A'C'(X' BX)^{-1} \end{aligned} \quad (144)$$

$$\begin{aligned}
\frac{\partial \operatorname{tr} \left( \mathbf{A}(\mathbf{X}\mathbf{B}\mathbf{X}')^{-1}\mathbf{C} \right)}{\partial \mathbf{X}} &= \frac{\partial \operatorname{tr} \left( \mathbf{C}\mathbf{A}(\mathbf{X}\mathbf{B}\mathbf{X}')^{-1} \right)}{\partial \mathbf{X}} = \frac{\partial \operatorname{tr} \left( (\mathbf{X}\mathbf{B}\mathbf{X}')^{-1}\mathbf{C}\mathbf{A} \right)}{\partial \mathbf{X}} \\
&= -(\mathbf{X}\mathbf{B}\mathbf{X}')^{-1}\mathbf{C}\mathbf{A}(\mathbf{X}\mathbf{B}\mathbf{X}')^{-1}\mathbf{X}\mathbf{B} \\
&\quad -(\mathbf{X}\mathbf{B}\mathbf{X}')^{-1'}\mathbf{A}'\mathbf{C}'(\mathbf{X}\mathbf{B}\mathbf{X}')^{-1'}\mathbf{X}\mathbf{B}' \quad (145)
\end{aligned}$$

$$\frac{\partial |\mathbf{A}\mathbf{X}\mathbf{B}|}{\partial \mathbf{X}} = \frac{\partial |\mathbf{B}'\mathbf{X}'\mathbf{A}'|}{\partial \mathbf{X}} = |\mathbf{A}\mathbf{X}\mathbf{B}|\mathbf{A}'(\mathbf{B}'\mathbf{X}'\mathbf{A}')^{-1}\mathbf{B}' \quad (146)$$

$$\begin{aligned}
\frac{\partial |\mathbf{A}\mathbf{X}'\mathbf{B}\mathbf{X}\mathbf{C}|}{\partial \mathbf{X}} &= |\mathbf{A}\mathbf{X}'\mathbf{B}\mathbf{X}\mathbf{C}| \times \\
&\quad \left( \mathbf{B}\mathbf{X}\mathbf{C}(\mathbf{A}\mathbf{X}'\mathbf{B}\mathbf{X}\mathbf{C})^{-1}\mathbf{A} + \mathbf{B}'\mathbf{X}\mathbf{A}'(\mathbf{A}\mathbf{X}'\mathbf{B}\mathbf{X}\mathbf{C})^{-1'}\mathbf{C}' \right) \quad (147)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial |\mathbf{A}\mathbf{X}\mathbf{B}\mathbf{X}'\mathbf{C}|}{\partial \mathbf{X}} &= |\mathbf{A}\mathbf{X}\mathbf{B}\mathbf{X}'\mathbf{C}| \times \\
&\quad \left( \mathbf{C}(\mathbf{A}\mathbf{X}\mathbf{B}\mathbf{X}'\mathbf{C})^{-1}\mathbf{A}\mathbf{X}\mathbf{B} + \mathbf{A}'(\mathbf{A}\mathbf{X}\mathbf{B}\mathbf{X}'\mathbf{C})^{-1'}\mathbf{C}'\mathbf{X}\mathbf{B}' \right) \quad (148)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial |\mathbf{A}(\mathbf{B}\mathbf{X}\mathbf{C})^{-1}\mathbf{D}|}{\partial \mathbf{X}} &= -|\mathbf{A}(\mathbf{B}\mathbf{X}\mathbf{C})^{-1}\mathbf{D}| \times \\
&\quad \left( \mathbf{C}(\mathbf{B}\mathbf{X}\mathbf{C})^{-1}\mathbf{D}(\mathbf{A}(\mathbf{B}\mathbf{X}\mathbf{C})^{-1}\mathbf{D})^{-1}\mathbf{A}\mathbf{B}\mathbf{X}\mathbf{C}^{-1}\mathbf{B} \right)' \quad (149)
\end{aligned}$$

$$\frac{\partial \lambda(\mathbf{A}\mathbf{X}\mathbf{B})}{\partial \mathbf{X}} = \frac{1}{q'p} \mathbf{A}'p\mathbf{q}'\mathbf{B}' \quad (151)$$

$$\frac{\partial \lambda(\mathbf{A}\mathbf{X}'\mathbf{B})}{\partial \mathbf{X}} = \frac{1}{q'p} \mathbf{B}p\mathbf{q}'\mathbf{A} \quad (152)$$

$$\frac{\partial \lambda(\mathbf{A}\mathbf{X}^{-1}\mathbf{B})}{\partial \mathbf{X}} = \frac{1}{q'p} - \mathbf{X}^{-1'}\mathbf{A}'p\mathbf{q}'\mathbf{B}'\mathbf{X}^{-1'} \quad (153)$$

$$\frac{\partial \lambda(\mathbf{A}\mathbf{X}'\mathbf{B}\mathbf{X}\mathbf{C})}{\partial \mathbf{X}} = \frac{1}{q'p} (\mathbf{B}\mathbf{X}\mathbf{C}p\mathbf{q}'\mathbf{A} + \mathbf{B}'\mathbf{X}\mathbf{A}'p\mathbf{q}'\mathbf{C}') \quad (154)$$

$$\frac{\partial \lambda(\mathbf{A}\mathbf{X}\mathbf{B}\mathbf{X}'\mathbf{C})}{\partial \mathbf{X}} = \frac{1}{q'p} (\mathbf{C}p\mathbf{q}'\mathbf{A}\mathbf{X}\mathbf{B} + \mathbf{A}'p\mathbf{q}'\mathbf{C}'\mathbf{X}\mathbf{B}') \quad (155)$$

なお、上記の固有値の微分に際して、 $\mathbf{q}$  ならびに  $\mathbf{p}$  は、固有値を求めるべき行列、たとえば  $\mathbf{A}\mathbf{X}\mathbf{B}$  の、その固有値に対応したの左右の固有ベクトルを表している。

$$\frac{\partial \operatorname{tr}(\mathbf{A} \operatorname{diag}(\mathbf{BXC})\mathbf{D})}{\partial \mathbf{X}} = (\mathbf{C} \operatorname{diag}(\mathbf{DA})\mathbf{B})' \quad (156)$$

$$\frac{\partial \operatorname{tr}(\mathbf{A} \operatorname{diag}(\mathbf{X}'\mathbf{BX})\mathbf{C})}{\partial \mathbf{X}} = \mathbf{BX} \operatorname{diag}(\mathbf{CA}) + (\operatorname{diag}(\mathbf{CA})\mathbf{X}'\mathbf{B})' \quad (157)$$

$$\frac{\partial \operatorname{tr}(\mathbf{A} \operatorname{diag}(\mathbf{BX}^{-1}\mathbf{C})\mathbf{D})}{\partial \mathbf{X}} = -(\mathbf{X}^{-1}\mathbf{C} \operatorname{diag}(\mathbf{DA})\mathbf{BX}^{-1})' \quad (158)$$

$$\frac{\partial \operatorname{tr}(\mathbf{A} \operatorname{diag}(\mathbf{X}'\mathbf{B}^{-1}\mathbf{X})\mathbf{C})}{\partial \mathbf{X}} = -\mathbf{BX}(\mathbf{X}'\mathbf{BX})^{-1} \operatorname{diag}(\mathbf{CA})(\mathbf{X}'\mathbf{BX})^{-1} \quad (159)$$

$$-((\mathbf{X}'\mathbf{BX})^{-1} \operatorname{diag}(\mathbf{CA})(\mathbf{X}'\mathbf{BX})^{-1}\mathbf{X}'\mathbf{B})' \quad (160)$$

$$\frac{\partial \operatorname{tr}(\mathbf{A}(\mathbf{X} \odot \mathbf{B})\mathbf{C})}{\partial \mathbf{X}} = \mathbf{B} \odot (\mathbf{CA})' \quad (161)$$

$$\frac{\partial \operatorname{tr}(\mathbf{A}(\mathbf{X}' \odot \mathbf{B})\mathbf{C})}{\partial \mathbf{X}} = \mathbf{B}' \odot (\mathbf{CA})' \quad (162)$$

$$\frac{\partial \operatorname{tr}(\mathbf{A}((\mathbf{X}'\mathbf{BX}) \odot \mathbf{C})\mathbf{D})}{\partial \mathbf{X}} = \mathbf{BX}(\mathbf{C}' \odot (\mathbf{DA})) + ((\mathbf{C}' \odot (\mathbf{DA}))\mathbf{X}'\mathbf{B})' \quad (163)$$

$$\frac{\partial \operatorname{tr}(\mathbf{A}(\mathbf{X}^{-1} \odot \mathbf{B})\mathbf{C})}{\partial \mathbf{X}} = -(\mathbf{X}^{-1}(\mathbf{B}' \odot (\mathbf{CA}))\mathbf{X}^{-1})' \quad (164)$$

$$\frac{\partial \operatorname{tr}(\mathbf{A}\mathbf{X}^{(p)}\mathbf{B})}{\partial \mathbf{X}} = p\mathbf{X}^{(p-1)} \odot (\mathbf{BA})' \quad (165)$$

$$\frac{\partial \operatorname{tr}(\mathbf{A}(\mathbf{X}'\mathbf{BX})^{(p)}\mathbf{C})}{\partial \mathbf{X}} = p(\mathbf{BX}((\mathbf{X}'\mathbf{BX})^{(p-1)'} \odot (\mathbf{CA})) \quad (166)$$

$$+ (((\mathbf{X}'\mathbf{BX})^{(p-1)'} \odot (\mathbf{CA}))\mathbf{X}'\mathbf{B})') \quad (167)$$

$$\frac{\partial \operatorname{tr}(\mathbf{A}\mathbf{X}^{(p)-1}\mathbf{B})}{\partial \mathbf{X}} = -p\mathbf{X}^{(p-1)} \odot (\mathbf{X}^{(p)-1}\mathbf{BAX}^{(p)-1})' \quad (168)$$

\*\*\*\*\*  
a matrix derivative of a scalar function of matrix/vector variables  
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In[636]= mD[tr[t[Y - X.B].W.(Y - X.B).V], B] // tf

Input = mD[ Tr[(Y - X.B)<sup>T</sup>.W.Y.V] - Tr[(Y - X.B)<sup>T</sup>.W.X.B.V], B]

Simplified Input =mD[ Tr[Y<sup>T</sup>.W.Y.V] - Tr[B<sup>T</sup>.X<sup>T</sup>.W.Y.V] - Tr[Y<sup>T</sup>.W.X.B.V] + Tr[B<sup>T</sup>.X<sup>T</sup>.W.X.B.V], B]

Rules Used : (1)A1((A+B).X) (2)S1(constant) (3)A1((A+B).X)

(4)A2(-A) (5)G3(A.t[X].B) (6)S2(tr[X.A])

(7)A1((A+B).X) (8)A2(-A) (9)G2(A.X.B)

(10)S2(tr[X.A]) (11)P1(tr[f.g]) (12)G2(A.X.B)

(13)S2(tr[X.A]) (14)G3(A.t[X].B) (15)S2(tr[X.A])

(1) mD0x (Tr[Y<sup>T</sup>.W.Y.V], B) + mD0x (-Tr[B<sup>T</sup>.X<sup>T</sup>.W.Y.V] - Tr[Y<sup>T</sup>.W.X.B.V] + Tr[B<sup>T</sup>.X<sup>T</sup>.W.X.B.V], B)

(2) Null

(3) mD0x (-Tr[B<sup>T</sup>.X<sup>T</sup>.W.Y.V], B) + mD0x (Tr[B<sup>T</sup>.X<sup>T</sup>.W.X.B.V] - Tr[Y<sup>T</sup>.W.X.B.V], B)

(4) -mD0x (Tr[B<sup>T</sup>.X<sup>T</sup>.W.Y.V], B)

(5) mD0x (Tr[B.V<sup>T</sup>.Y<sup>T</sup>.W<sup>T</sup>.X], B)

(6) (V<sup>T</sup>.Y<sup>T</sup>.W<sup>T</sup>.X)<sup>T</sup>

(7) mD0x (-Tr[Y<sup>T</sup>.W.X.B.V], B) + mD0x (Tr[B<sup>T</sup>.X<sup>T</sup>.W.X.B.V], B)

(8) -mD0x (Tr[Y<sup>T</sup>.W.X.B.V], B)

(9) mD0x (Tr[B.V.Y<sup>T</sup>.W.X], B)

(10) (V.Y<sup>T</sup>.W.X)<sup>T</sup>

(11) mD0x (Tr[B<sup>T</sup>.Q8\$7248], B) + mD0x (Tr[Q9\$7248.X<sup>T</sup>.W.X.B.V], B)

(12) mD0x (Tr[B.V.Q9\$7248.X<sup>T</sup>.W.X], B)

(13) (V.Q9\$7248.X<sup>T</sup>.W.X)<sup>T</sup>

(14) mD0x (Tr[B.Q8\$7248<sup>T</sup>], B)

(15) Q8\$7248<sup>TT</sup>

(15) (V.B<sup>T</sup>.X<sup>T</sup>.W.X)<sup>T</sup> + (X<sup>T</sup>.W.X.B.V)<sup>TT</sup>

Out[636]/TraditionalForm=

$$-X^T.W.Y.V + X^T.W.X.B.V - (V.Y^T.W.X)^T + (V.B^T.X^T.W.X)^T$$

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In[526]=  $\Sigma = \Lambda \cdot \Xi \cdot t[\Lambda] + \Psi$ 
           $mD[\text{tr}[S \cdot \text{Inverse}[\Sigma]] - \text{Log}[\text{Det}[S \cdot \Sigma]], \Lambda] // \text{tf}$ 

Out[526]=  $\Psi + \Lambda \cdot \Xi \cdot \text{Transpose}[\Lambda]$ 

Input =  $mD[\text{Tr}[S \cdot (\Psi + \Lambda \cdot \Xi \cdot \Lambda^T)^{-1}] - \log(\{S \cdot (\Psi + \Lambda \cdot \Xi \cdot \Lambda^T)\}), \Lambda]$ 

Simplified Input =  $mD[\text{Tr}[S \cdot (\Psi + \Lambda \cdot \Xi \cdot \Lambda^T)^{-1}] - \log(\{S \cdot \Psi + S \cdot \Lambda \cdot \Xi \cdot \Lambda^T\}), \Lambda]$ 

Rules Used : (1)A1((A+B).X) (2)A2(-A) (3)A4(Log f[X])
(4)C2(det) (5)A1((A+B).X) (6)S1(constant)
(7)G2(A.X.B) (8)P1(tr[f.g]) (9)G3(A.t[X].B)
(10)S2(tr[X.A]) (11)S2(tr[X.A]) (12)G4.1(A.inv[B].C)
(13)C1.1(inv) (14)A1((A+B).X) (15)S1(constant)
(16)G2(A.X.B) (17)P1(tr[f.g]) (18)G3(A.t[X].B)
(19)S2(tr[X.A]) (20)S2(tr[X.A])

(1)  $mD0x(-\log(\{S \cdot \Psi + S \cdot \Lambda \cdot \Xi \cdot \Lambda^T\}), \Lambda) + mD0x(\text{Tr}[S \cdot (\Psi + \Lambda \cdot \Xi \cdot \Lambda^T)^{-1}], \Lambda)$ 
(2)  $-mD0x(\log(\{S \cdot \Psi + S \cdot \Lambda \cdot \Xi \cdot \Lambda^T\}), \Lambda)$ 
(3)  $\frac{mD0x(\{S \cdot \Psi + S \cdot \Lambda \cdot \Xi \cdot \Lambda^T\}, \Lambda)}{\{S \cdot \Psi + S \cdot \Lambda \cdot \Xi \cdot \Lambda^T\}}$ 
(4)  $mD0x(\text{Tr}[(\{Q9\6920 \mid Q9\6920^{T-1}\}^T \cdot S \cdot \Psi) + \text{Tr}[(\{Q9\6920 \mid Q9\6920^{T-1}\}^T \cdot S \cdot \Lambda \cdot \Xi \cdot \Lambda^T)], \Lambda)$ 
(5)  $mD0x(\text{Tr}[(\{Q9\6920 \mid Q9\6920^{T-1}\}^T \cdot S \cdot \Psi), \Lambda] + mD0x(\text{Tr}[(\{Q9\6920 \mid Q9\6920^{T-1}\}^T \cdot S \cdot \Lambda \cdot \Xi \cdot \Lambda^T)], \Lambda)$ 
(6) Null
(7)  $mD0x(\text{Tr}[\Lambda \cdot \Xi \cdot \Lambda^T \cdot (\{Q9\6920 \mid Q9\6920^{T-1}\}^T \cdot S), \Lambda])$ 
(8)  $mD0x(\text{Tr}[\Lambda \cdot Q8\6942], \Lambda) + mD0x(\text{Tr}[Q8\6942 \cdot \Xi \cdot \Lambda^T \cdot (\{Q9\6920 \mid Q9\6920^{T-1}\}^T \cdot S)], \Lambda)$ 
(9)  $mD0x(\text{Tr}[\Lambda \cdot \Xi^T \cdot Q8\6942^T \cdot S^T \cdot (\{Q9\6920 \mid Q9\6920^{T-1}\}], \Lambda)$ 
(10)  $(\Xi^T \cdot Q8\6942^T \cdot S^T \cdot (\{Q9\6920 \mid Q9\6920^{T-1}\})^T$ 
(11)  $Q8\6942^T$ 
(11)  $(\Xi \cdot \Lambda^T \cdot (\{Q9\6920 \mid Q9\6920^{T-1}\}^T \cdot S)^T + (\Xi^T \cdot \Lambda^T \cdot S^T \cdot (\{Q9\6920 \mid Q9\6920^{T-1}\})^T$ 
(11)  $(\Xi \cdot \Lambda^T \cdot (\{S \cdot \Psi + S \cdot \Lambda \cdot \Xi \cdot \Lambda^T\} (S \cdot \Psi + S \cdot \Lambda \cdot \Xi \cdot \Lambda^T)^{T-1})^T \cdot S)^T + (\Xi^T \cdot \Lambda^T \cdot S^T \cdot (\{S \cdot \Psi + S \cdot \Lambda \cdot \Xi \cdot \Lambda^T\} (S \cdot \Psi + S \cdot \Lambda \cdot \Xi \cdot \Lambda^T)^{T-1}))^T$ 
(12)  $mD0x(\text{Tr}[(\Psi + \Lambda \cdot \Xi \cdot \Lambda^T)^{-1} \cdot S], \Lambda)$ 
(13)  $-mD0x(\text{Tr}[Q9\6957^{-1} \cdot S \cdot Q9\6957^{-1} \cdot \Psi] + \text{Tr}[Q9\6957^{-1} \cdot S \cdot Q9\6957^{-1} \cdot \Lambda \cdot \Xi \cdot \Lambda^T], \Lambda)$ 
(14)  $mD0x(\text{Tr}[Q9\6957^{-1} \cdot S \cdot Q9\6957^{-1} \cdot \Psi], \Lambda) + mD0x(\text{Tr}[Q9\6957^{-1} \cdot S \cdot Q9\6957^{-1} \cdot \Lambda \cdot \Xi \cdot \Lambda^T], \Lambda)$ 
(15) Null
(16)  $mD0x(\text{Tr}[\Lambda \cdot \Xi \cdot \Lambda^T \cdot Q9\6957^{-1} \cdot S \cdot Q9\6957^{-1}], \Lambda)$ 
(17)  $mD0x(\text{Tr}[\Lambda \cdot Q8\6971], \Lambda) + mD0x(\text{Tr}[Q8\6971 \cdot \Xi \cdot \Lambda^T \cdot Q9\6957^{-1} \cdot S \cdot Q9\6957^{-1}], \Lambda)$ 
(18)  $mD0x(\text{Tr}[\Lambda \cdot \Xi^T \cdot Q8\6971^T \cdot Q9\6957^{-1T} \cdot S^T \cdot Q9\6957^{-1T}], \Lambda)$ 
(19)  $(\Xi^T \cdot Q8\6971^T \cdot Q9\6957^{-1T} \cdot S^T \cdot Q9\6957^{-1T})^T$ 
(20)  $Q8\6971^T$ 
(20)  $(\Xi \cdot \Lambda^T \cdot Q9\6957^{-1} \cdot S \cdot Q9\6957^{-1})^T + (\Xi^T \cdot \Lambda^T \cdot Q9\6957^{-1T} \cdot S^T \cdot Q9\6957^{-1T})^T$ 
(20)  $-(\Xi \cdot \Lambda^T \cdot (\Psi + \Lambda \cdot \Xi \cdot \Lambda^T)^{-1} \cdot S \cdot (\Psi + \Lambda \cdot \Xi \cdot \Lambda^T)^{-1})^T - (\Xi^T \cdot \Lambda^T \cdot (\Psi + \Lambda \cdot \Xi \cdot \Lambda^T)^{-1T} \cdot S^T \cdot (\Psi + \Lambda \cdot \Xi \cdot \Lambda^T)^{-1T})^T$ 

Out[527]/TraditionalForm=
 $-(S \cdot \Psi + S \cdot \Lambda \cdot \Phi \cdot \Lambda^T)^{-1} \cdot S \cdot \Lambda \cdot \Phi - (\Psi + \Lambda \cdot \Phi \cdot \Lambda^T)^{-1} \cdot S \cdot (\Psi + \Lambda \cdot \Phi \cdot \Lambda^T)^{-1} \cdot \Lambda \cdot \Phi -$ 
 $(\Phi \cdot \Lambda^T \cdot (S \cdot \Psi + S \cdot \Lambda \cdot \Phi \cdot \Lambda^T)^{-1} \cdot S)^T - (\Phi \cdot \Lambda^T \cdot (\Psi + \Lambda \cdot \Phi \cdot \Lambda^T)^{-1} \cdot S \cdot (\Psi + \Lambda \cdot \Phi \cdot \Lambda^T)^{-1})^T$ 

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