Introduction to Bayesian Statistics

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Outline

- 1. Introduction to Bayes Theorem
- 2. Binomial Model and Poisson Model
- 3. Normal Model
- 4. Normal Regression Model
- 5. Bayesian Numerical Calculations (Markov Chain Monte Carlo)
- 6. Application of MCMC
- 7. Model Selection
- 8. miscellaneous topics

Multiple Independent Observations

 If two events are mutually independent the joint probability of two events can be written as

> $\Pr(A_{i1} \& A_{i2}|B_j) = \Pr(A_{i1}, A_{i2}|B_j)$ = $\Pr(A_{i1}|B_j) \times \Pr(A_{i2}|B_j)$

Multiple Independent Observations

CPT	or Model Probab	oility	$\Pr(Ai1,Ai2 \mid Bj) =$	Pr(A	Ai1 Bj) Pr(Ai2 Bj)
	B_1		B_j		B_b
A_1, A_1	$\Pr\left(A_1 B_1\right)\Pr\left(A_1 B_1\right)$		$\Pr\left(A_1 B_j\right)\Pr\left(A_1 B_j\right)$		$\Pr\left(A_1 B_b\right)\Pr\left(A_1 B_b\right)$
A_1, A_2	$\Pr\left(A_1 B_1\right)\Pr\left(A_2 B_1\right)$		$\Pr\left(A_1 B_j\right)\Pr\left(A_2 B_j\right)$		$\Pr\left(A_1 B_b\right)\Pr\left(A_2 B_b\right)$
÷	•	:		:	:
A_{i1}, A_{i2}	$\Pr\left(A_{i1} B_1\right)\Pr\left(A_{i2} B_1\right)$		$\Pr\left(A_{i1} B_j\right)\Pr\left(A_{i2} B_j\right)$		$\Pr\left(A_{i1} B_b\right)\Pr\left(A_{i2} B_b\right)$
÷		:		:	:
A_a, A_a	$\Pr\left(A_a B_1\right)\Pr\left(A_a B_1\right)$		$\Pr\left(A_a B_j\right)\Pr\left(A_a B_j\right)$		$\Pr\left(A_a B_b\right)\Pr\left(A_a B_b\right)$
$\Pr\left(B\right)$	$\Pr(B_1)$		$\Pr\left(B_{j} ight)$		$\Pr\left(B_b\right)$

If $A_i, i = 1, 2, ..., a$ are mutually exclusive and exhaustive

$$\sum_{i1=1}^{a} \sum_{i2=1}^{a} \Pr(A_{i1}, A_{i2} | B_j) = 1$$

Joint Probability

$$\Pr(A_{i1}, A_{i2}, B_j) = \Pr(A_{i1}, A_{i2}|B_j) \times \Pr(B_j)$$

Posterior Probability

$$\Pr(B_j | A_{i1}, A_{i2}) = \frac{1}{\sum_{j=1}^b \Pr(A_{i1}, A_{i2}, B_j)} \Pr(A_{i1}, A_{i2}, B_j)$$

3

Multiple Independent Observations

Joint Probability Table

 $\Pr\left(A_{i1}, A_{i2}, B_{j}\right) = \Pr\left(A_{i1}, A_{i2} | B_{j}\right) \times \Pr\left(B_{j}\right)$

	B1		Bj		Bb	$\Pr(A, A)$
A_1, A_1	$\Pr\left(A_1 B_1\right)\Pr\left(A_1 B_1\right)\Pr\left(B_1\right)$		$\Pr\left(A_1 B_j\right)\Pr\left(A_1 B_j\right)\Pr\left(B_j\right)$		$\Pr\left(A_1 B_b\right)\Pr\left(A_1 B_b\right)\Pr\left(B_j\right)$	$\Pr\left(A_1,A_1\right)$
A_1, A_2	$\Pr{(A_1 B_1)}\Pr{(A_2 B_1)}\Pr{(B_1)}$		$\Pr\left(A_1 B_j\right)\Pr\left(A_2 B_j\right)\Pr\left(B_j\right)$		$\Pr(A_1 B_b)\Pr(A_2 B_b)\Pr(B_j)$	$\Pr\left(A_1,A_2\right)$
:		:	:	:		:
A _{i1} , A _{i2}	$\Pr\left(A_{i1} B_1\right)\Pr\left(A_{i2} B_1\right)\Pr\left(B_1\right)$		$\Pr\left(A_{i1} B_j\right)\Pr\left(A_{i2} B_j\right)\Pr\left(B_j\right)$		$\Pr\left(A_{i1} B_b\right)\Pr\left(A_{i2} B_b\right)\Pr\left(B_j\right)$	$\Pr\left(A_{i1}, A_{i2}\right)$
						(*1' **)
:	:	1		:		:
A_a, A_a	$\Pr{(A_a B_1)}\Pr{(A_a B_1)}\Pr{(B_1)}$		$\Pr\left(A_{a} B_{j}\right)\Pr\left(A_{a} B_{j}\right)\Pr\left(B_{j}\right)$		$\Pr\left(A_{a} B_{b}\right)\Pr\left(A_{a} B_{b}\right)\Pr\left(B_{j}\right)$	$\Pr\left(A_a, A_a\right)$
$\Pr(B)$	$\Pr(B_1)$		$\Pr(B_j)$		$Pr(B_b)$	

Posterior Probability

$$\Pr(B_j|A_{i1}, A_{i2}) = \frac{1}{\sum_{j=1}^{b} \Pr(A_{i1}, A_{i2}, B_j)} \Pr(A_{i1}, A_{i2}, B_j)$$

5

Coin Tossed Twice

CPT of A given B and Marginal of B	Joint Probability of A and B and marginal of A	Posterior Probability
B1:1/2 B2:2/3	B1:1/2 B2:2/3 Am	B1:1/2 B2:2/3 sum
A1:HH .25000 .44444	A1:HH .12500 .22222 .34722	A1:HH .36000 .64000 1.
A2:HT .25000 .22222	A2:HT .12500 .11111 .23611	A2:HT .52941 .47059 1.
A3:TH .25000 .22222	A3:TH .12500 .11111 .23611	A3:TH .52941 .47059 1.
A4:TT .25000 .11111	A4:TT .12500 .05556 .18056	A4:TT .69231 .30769 1.
Bm .50000 .50000	Bm .50000 .50000 1.0000	Bm .50000 .50000

Note that the posterior odds of not Fair to Fair, given Head, is larger with this example.

 $\frac{\Pr\left(\textit{notFair}|\textit{Head}\right)}{\Pr\left(\textit{Fair}|\textit{Head}\right)} = \frac{.571}{.429} = 1.333 < 1.778 = \frac{.640}{.360} = \frac{\Pr\left(\textit{notFair}|\textit{Head},\textit{Head}\right)}{\Pr\left(\textit{Fair}|\textit{Head},\textit{Head}\right)}$

Multiple observations provide more information on B.

Multiple Independent Observations

Posterior Probability

$$\Pr(B_j | A_{i1}, A_{i2}) = \frac{1}{\sum_{j=1}^{b} \Pr(A_{i1}, A_{i2}, B_j)} \Pr(A_{i1}, A_{i2}, B_j)$$

	B_1		B_j		B_b	$\Pr\left(A_{i1}, A_{i2}\right)$
A_1, A_1	$\frac{\Pr\left(A_1 B_1\right)\Pr\left(A_1 B_1\right)\Pr\left(B_1\right)}{\Pr\left(A_1,A_1\right)}$		$\frac{\Pr\left(A_1 B_j\right)\Pr\left(A_1 B_j\right)\Pr\left(B_j\right)}{\Pr\left(A_1,A_1\right)}$		$\frac{\Pr\left(A_1 B_b\right)\Pr\left(A_1 B_b\right)\Pr\left(B_j\right)}{\Pr\left(A_1,A_1\right)}$	$\Pr\left(A_1,A_1\right)$
A_1, A_2	$\frac{\Pr\left(A_1 B_1\right)\Pr\left(A_2 B_1\right)\Pr\left(B_1\right)}{\Pr\left(A_1,A_2\right)}$		$\frac{\Pr\left(A_1 B_j\right)\Pr\left(A_2 B_j\right)\Pr\left(B_j\right)}{\Pr\left(A_1,A_2\right)}$		$\frac{\Pr\left(A_1 B_b\right)\Pr\left(A_2 B_b\right)\Pr\left(B_j\right)}{\Pr\left(A_1,A_2\right)}$	$\Pr\left(A_1,A_2\right)$
÷		:		:		
A_{i1}, A_{i2}	$\frac{\Pr\left(A_{i1} B_1\right)\Pr\left(A_{i2} B_1\right)\Pr\left(B_1\right)}{\Pr\left(A_{i1},A_{i2}\right)}$		$\frac{\Pr\left(A_{i1} B_j\right)\Pr\left(A_{i2} B_j\right)\Pr\left(B_j\right)}{\Pr\left(A_{i1},A_{i2}\right)}$		$\frac{\Pr\left(A_{i1} B_b\right)\Pr\left(A_{i2} B_b\right)\Pr\left(B_j\right)}{\Pr\left(A_{i1},A_{i2}\right)}$	$\Pr\left(A_{i1}, A_{i2}\right)$
				1		
A_a, A_a	$\frac{\Pr\left(A_{a} B_{1}\right)\Pr\left(A_{a} B_{1}\right)\Pr\left(B_{1}\right)}{\Pr\left(A_{a},A_{a}\right)}$	220	$\frac{\Pr\left(A_{a} B_{j}\right)\Pr\left(A_{a} B_{j}\right)\Pr\left(B_{j}\right)}{\Pr\left(A_{a},A_{a}\right)}$		$\frac{\Pr\left(A_{a} B_{b}\right)\Pr\left(A_{a} B_{b}\right)\Pr\left(B_{j}\right)}{\Pr\left(A_{a},A_{a}\right)}$	$\Pr\left(A_a, A_a\right)$
$\Pr(B)$	$\Pr(B_1)$		$\Pr\left(B_{j}\right)$		$\Pr\left(B_{b}\right)$	

Coin Tossed Twice

CPT of A given B and Marginal of B	Joint Probability of A and B and marginal of A	Posterior Probability
B1:1/2 B2:2/3	B1:1/2 B2:2/3 Am	B1:1/2 B2:2/3 sum
A1:HH .25000 .44444	A1:HH .12500 .22222 .34722	A1:HH .36000 .64000 1.
A2:HT .25000 .22222	A2:HT .12500 .11111 .23611	A2:HT .52941 .47059 1.
A3:TH .25000 .22222	A3:TH .12500 .11111 .23611	A3:TH .52941 .47059 1.
A4:TT .25000 .11111	A4:TT .12500 .05556 .18056	A4:TT .69231 .30769 1.
Bm .50000 .50000	Bm .50000 .50000 1.0000	Bm .50000 .50000

Note also that, $\Pr(Fair|Head, Tail) = \Pr(Fair|Tail, Head)$. Number of Heads is sufficient for the inference.

Two Test Scores

Conditional Probability of A Joint Probability of A Description PROF Birl D	Test So	core (tw	o score	s)										
PROB1 B1:L B2:M B3:H PROB2 B1:L B2:M B3:H Am PROB3 B1:L B2:M B3:H A1:00 .01000 .00000 A1:00 .33333 .66667 .00000 A2:01 .00000 .00200 .00000 .01250 .41:00 .33333 .66667 .00000 A3:02 .04000 .02000 .00000 .42:01 .60000 .00000 .02000 .42:00 .60000 .00000 .02000 .42:00 .60000 .00000 .42:00 .60000 .00000 .44:03 .00000 .00000 .44:03 .00000 .00000 .44:03 .00000 .00000 .44:03 .00000 .00000 .46:05 .00000 .00000 .46:05 .00000 .00000 .46:05 .00000 .46:05 .00000 .46:05 .00000 .46:05 .00000 .46:05 .00000 .46:05 .00000 .41:14 .0000 .40:00 .00000 .41:14 .0000 .40:0	Conditi	ional Pr	robabili	ty of A	Joint H	robabil	ity of	A and I	в					
A1:00 .01000 .00000 A1:00 .00250 .00500 .00750 A1:00 .3333 .66667 .00000 A2:01 .03000 .00000 A2:01 .00000 .02500 .60000 .40000 .00000 A3:02 .04000 .03000 .00000 .41:03 .50000 .60000 .60000 .60000 .61:05 .00000 .61:05 .00000 .60000 .61:05 .00000 .60000 .00000 .60000 .00000 .60000 .00000 .60000 .00000 .00000 .00000 .60000 .00000 .60000 .00000 .60000 .00000 .60000 .00000 .60000 .00000 .00000 .00000 .00000 .60000 .00000 .60000 .00000 .60000 .00000 .60000 .01111 .00000 .00000 .010	given H	3 (Model	L)		and the	• Margir	al			Posterio	or Proba	ability		
A2:01 .03000 .00000 A2:01 .00750 .00000 A2:01 .60000 .40000 .00000 A3:02 .04000 .03000 .00000 A3:02 .50000 .50000 .60000 .60000 .40000 .00000 A3:02 .50000 .60000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .00000 .6111 .0000 .00000 .00000 .01126 .47:10 .60000 .00000 .00000 .01126 .47:10 .60000 .00000 .00000 .01215 .00000 .00000 .01126 .47:10 .60000 .00000 .01215 .00000 .00000 .01126 .47:11 .00000 .00000 .01126 .47:11 .00000 .00000 .1111 .00000 .00000 <	PROB1	B1:L	B2:M	B3:H	PROB2	B1:L	B2:M	B3:H	Am	PROB3	B1:L	B2:M	B3:H	
A2:01 .03000 .00000 A2:01 .00750 .00000 A2:01 .60000 .40000 .00000 A3:02 .04000 .03000 .00000 A3:02 .50000 .50000 .60000 .60000 .40000 .00000 A3:02 .50000 .60000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .6100 .00000 .00000 .6111 .0000 .00000 .00000 .01126 .47:10 .60000 .00000 .00000 .01126 .47:10 .60000 .00000 .00000 .01215 .00000 .00000 .01126 .47:10 .60000 .00000 .01215 .00000 .00000 .01126 .47:11 .00000 .00000 .01126 .47:11 .00000 .00000 .1111 .00000 .00000 <														
A3:02 .04000 .02000 .02000 A3:02 .50000 .62000 .62000 .62000 .62000 .62000 .02000 A4:03 .25000 .75000 .00000 A5:04 .00000 .02000 A5:04 .00000 .01000 .00000 A4:03 .00000 .00000 A5:04 .00000 .00000 A6:05 .00000 .00000 A6:00 .00000 A6:11 .00000 A6:00 .00000 A1:11 .00000 A6:00 .00000 A1:11 .00000 .00000 A1:11 .00000 A1:11 .00000 .00000 A1:11 .00000 A1:11 .00000 .00000 A1:11 .00000 .00000 A1:11 .00000 .00000 A1:11 .00000 .00000 .00000 A1:1	A1:00	.01000	.01000	.00000	A1:00	.00250	.00500	.00000	.00750	A1:00	.33333	.66667	.00000	
A4:03 .02000 .03000 .00000 A4:03 .02000 .02000 A4:03 .25000 .75000 .00000 A5:04 .00000 .01000 .00000 A5:04 .00000 .11114 .00000 .00000 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .00000 .00000 .11114 .00000 .00000 .00000 .00000 .11114 .00000 .00000 .00000 .11114 .00000 .00000 .01000 .00000 .111114 .00000 .00000 .0	A2:01	.03000	.01000	.00000	A2:01	.00750	.00500	.00000	.01250	A2:01	.60000	.40000	.00000	
A5:04 .00000 .02000 A5:04 .00000 .01000 A5:04 .00000 .01000 A5:04 .00000 .00000 A5:04 .00000 .00000 A5:05 .00000 .00000 A5:05 .00000 .00000 A5:05 .00000 .00000 A5:05 .00000 .00000 A5:01 .00000 .00000 .00000 A5:11 .00000 .00000 A5:11 .00000 .00000 A5:11 .00000 .00000 A5:11 .50000 .00000 .01150 .00000 .01150 .00000 .11:14 .00000 .00000 .11:14 .00000 .00000 .11:14 .00000 .00000 .11:14 .00000 .00000 .11:14 .00000 .00000 .11:14 .00000 .00000 .11:14 .00000 .00000 .11:14 .00000 .00000 .11:14 .00000 .00000 .11:14 .00000 .00000 .11:14 .00000 .00000 .11:14 .00000 .00000 .11:14 .00000 .00000 .11:14 .00000 .00000 .11:14 .00000 .00000 .11:1	A3:02	.04000	.02000	.00000	A3:02	.01000	.01000	.00000	.02000	A3:02	.50000	.50000	.00000	
A6:05 .00000 .01000 .00000 A6:05 .00000 A6:05 .00000 .00000 A6:05 .00000 .00000 A7:10 .00000 .00000 A7:11 .8111 .8112 .00000 .00000 A1:11 .00000 .00000 A1:112	A4:03	.02000	.03000	.00000	A4:03	.00500	.01500	.00000	.02000	A4:03	.25000	.75000	.00000	
A7:10 .03000 .01000 .00000 A7:10 .000750 .00000 .021250 A7:10 .60000 .60000 .00000 A9:11 .02000 .00000 A9:11 .02000 .00000 A9:11 .03000 .00000 A9:11 .75000 .25000 .00000 A10:13 .05000 .00000 A10:13 .00000 .01000 .01000 .00000 .11:14 .00000 .00000 .01000 .01000 .00000 .11:14 .00000 .00000 .01000 .01000 .00000 .01000 .00000 .01000 .00000 .01000 .00000 .01000 .00000 .01000 .00000 .011:14 .00000 .00000 .00000 .011:14 .00000 .00000 .011:14 .00000 .00000 .00000 .02200 .00000 .011:14 .00000 .00000 .011:14 .00000 .00000 .02000 .00000 .02000 .00000 .02000 .00000 .02000 .00000 .02000 .00000 .02000 .02000 .00000 .02000 .00000 .00000 .02000	A5:04	.00000	.02000	.00000	A5:04	.00000	.01000	.00000	.01000	A5:04	.00000	1.0000	.00000	
A8:11 .09000 .01000 .00000 A8:11 .02250 .00500 .00000 .02750 A8:11 .8111 .8112 .01000 A9:12 .12000 .00000 .40000 .01000 .00000 .40010 .75000 .55000 .00000 A11:14 .00000 .02000 .00000 A11:14 .00000 .01000 .00000 .01113 .50000 .50000 .00000 A11:14 .00000 .02000 .00000 A11:14 .00000 .00000 A11:14 .00000 .00000 .00000 .00000 .00000 .00000 A11:11 .50000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .11114 .00000 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .11114 .00000 .00000 .11114 .10000 .00000 .11114 .10000 .00000 .11114 .100000 .00000 .1111	A6:05	.00000	.01000	.00000	A6:05	.00000	.00500	.00000	.00500	A6:05	.00000	1.0000	.00000	
A9:12 12000 .02000 .00000 A9:12 .75000 .25000 .00000 A10:13 .05000 .03000 .00000 .10113 .01500 .00000 .11114 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .00000 .01100 .00000 .00000 .11114 .00000 .00000 .011114 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .00000 .11114 .00000 .01000 .00000 .11114 .00000 .01100 .00000 .11114 .00000 .01000 .00000 .11114 .00000 .01000 .01000 .00000 .11114 .00000	A7:10	.03000	.01000	.00000	A7:10	.00750	.00500	.00000	.01250	A7:10	.60000	.40000	.00000	
A10:13 .06000 .03000 .00000 A10:13 .05000 .00000 A10:14 .00000 .00000 A10:14 .00000 .00000 A11:14 .00000 .00000 A11:12 .00000 .00000 .00000 A11:12 .00000 .2000 .00000 .2000 .411:13 .00000 .2000 .411:13 .00000 .20000 .2000 .411:13 .00000 .20000 .20000 .20000 .20000 .20000 .20000 .20000 .20000 .20000 .20000 .20000 .20000 .20000 .20000 .20000 .20000	A8:11	.09000	.01000	.00000	A8:11	.02250	.00500	.00000	.02750	A8:11	.81818	.18182	.00000	
A11:14 .00000 .02000 .02000 A11:14 .00000 A11:14 .00000 .00000 A11:14 .00000 .00000 A12:15 .00000 .00000 A12:17 .50000 .00000 .00000 A14:11 .50000 .00000 .00000 A12:15 .00000 .00000 A14:11 .50000 .00000 .00000 A14:11 .50000 .00000 .00000 A14:11 .50000 .00000 .00000 A14:11 .50000 .00000 A14:11 .50000 .00000 A14:11 .50000 .00000 A14:11 .50000 .00000 A14:11 .00000 .00000 A14:11 .00000 .00000 A14:11 .00000 .00000 .111:14 .00000 .00000 .00000 .111:14 .00000 .00000 .111:14 .00000 .00000 .111:14 .00000 .00000 .01100 .00200 <td>A9:12</td> <td>.12000</td> <td>.02000</td> <td>.00000</td> <td>A9:12</td> <td>.03000</td> <td>.01000</td> <td>.00000</td> <td>.04000</td> <td>A9:12</td> <td>.75000</td> <td>.25000</td> <td>.00000</td> <td></td>	A9:12	.12000	.02000	.00000	A9:12	.03000	.01000	.00000	.04000	A9:12	.75000	.25000	.00000	
A12:15 00000 .01000 .00000 A12:15 .00000 A12:15 .00000 .00000 A12:15 .00000 .00000 A12:15 .00000 .00000 .00000 A13:20 .00000 .01000 .00000 .02000 .00000 A13:20 .00000 .01000 .00000 .02000 .02000 .02000 .02000 .02000 .021:15 .00000 .02000 .02500 .02000 .02500 .00000 .04000 .02000 .03000 .00500 .05500 .17124 .00000 .66667 .33333 A18:25 .00000 .01000 .05500 .02000 .131:30 .0500 .00000 .20001 .20000 .00000 .20131 .50000 .5000 .20131 .50000 .5000 .221:32 .50000 .50000 .20131 .50000 .20001 .20000	A10:13	.06000	.03000	.00000	A10:13	.01500	.01500	.00000	.03000	A10:13	.50000	.50000	.00000	
A13:20 .04000 .02000 .00000 A13:20 .01000 .00000 .02000 .60000 .615:22 .64000 .415:22 .64000 .415:22 .64000 .415:22 .64000 .415:22 .64000 .415:22 .64000 .415:23 .62000 .65500 .415:22 .64000 .42000 .42000 .62000 .62000 .415:12 .00000 .415:12 .00000 .415:12 .00000 .415:12 .00000 .415:12 .00000 .415:12 .00000 .415:12 .00000 .415:12 .00000 .415:12 .00000 .2000 .415:12 .00000 .20000 .415:12 .00000 .20000 .415:12 .00000 .20000 .415:12 .00000 .20000 .415:12 .00000 .20000	A11:14	.00000	.02000	.00000	A11:14	.00000	.01000	.00000	.01000	A11:14	.00000	1.0000	.00000	
A14:21 12000 .02000 .00000 A14:21 .75000 .25000 .00000 A15:22 .16000 .04000 .01000 .02000 .02000 .02550 .415:22 .64000 .32000 .44000 A15:22 .06000 .02000 .02000 .00250 .02500 .415:22 .64000 .32000 .44000 A15:23 .08000 .04000 .416:23 .02000 .02000 .03000 .417:24 .00000 .66647 .33333 A18:25 .00000 .02000 .03000 .01050 .01050 .02000 .418:23 .56000 .57143 .42857 A19:30 .02000 .02000 .03000 .00000 .22000 .25000 .75000 .00000 A21:32 .08000 .06000 .421:32 .02000 .03000 .420:31 .55000 .60000 .420:31 .15385 .233:4 .00000 .420:31 .55000 .421:32 .00000 .420:000 .420:000 .420:000 .420:000 .420:000 .420:000 .420:0000 .420:0000 .420	A12:15	.00000	.01000	.00000	A12:15	.00000	.00500	.00000	.00500	A12:15	.00000	1.0000	.00000	
A15:22 16000 .04000 .01000 A15:22 .04000 .02050 .02550 A15:22 .64000 .32000 .04000 A15:22 .08000 .04000 A17:24 .00000 .03000 .05500 .05500 A15:23 .33345 .4545 .09091 A17:24 .00000 .04000 A17:24 .00000 .03000 .03000 .03000 A17:24 .00000 .66667 .33333 A18:25 .00000 .03000 .03000 .03000 .03000 .02000 .412:35 .00000 .32030 .420:31 .50000 .50000 .60000 .02000 .421:32 .82000 .60000 .02000 .421:32 .20000 .60000 .221:33 .15000 .50000 .60000 .02000 .221:33 .15100 .03000 .05500 .421:33 .60000 .60000 .60000 .20000 .421:32 .20000 .60500 .421:33 .60000 .20000 .421:33 .40000 .60000 .421:33 .60000 .20000 .40000 .421:33 .60000 .221:33 .15355	A13:20	.04000	.02000	.00000	A13:20	.01000	.01000	.00000	.02000	A13:20	.50000	.50000	.00000	
A16:23 .08000 .06000 .02000 A16:23 .02000 .03000 .00500 .05500 A16:23 .32364 .54545 .09091 A17:24 .00000 .02000 .03000 .02000 .03000 .0750 .01500 .01500 .6667 .33333 A18:25 .00000 .02000 .03000 A17:24 .00000 .02000 .11512 .00000 .6667 .33333 A18:30 .02000 .03000 .00000 A12:31 .01500 .00000 .22000 .25000 .75000 .00000 A22:31 .06000 .02000 .421:32 .02000 .03000 .00000 .22000 .23031 .15386 .22333 .15886 .69231 .15385 A22:33 .00000 .02000 .421:32 .02000 .03000 .22331 .00000 .60000 .420:33 .00000 .60000 .420:35 .00000 .420:35 .00000 .420:35 .00000 .420:35 .00000 .420:35 .00000 .420:35 .00000 .420:45 .00000 .420:45 <t< td=""><td>A14:21</td><td>.12000</td><td>.02000</td><td>.00000</td><td>A14:21</td><td>.03000</td><td>.01000</td><td>.00000</td><td>.04000</td><td>A14:21</td><td>.75000</td><td>.25000</td><td>.00000</td><td></td></t<>	A14:21	.12000	.02000	.00000	A14:21	.03000	.01000	.00000	.04000	A14:21	.75000	.25000	.00000	
A17:24 .00000 .04000 A17:24 .00000 .02000 .33333 A18:25 .00000 .02000 .03000 .01000 .02000 .33333 A19:30 .02000 .03000 .00000 .419:30 .00000 .02000 .419:30 .00000 .22000 .419:30 .20000 .20150 .00000 .221:31 .50000 .50000 .00000 .221:31 .50000 .50000 .20000 .421:35 .50000 .60000 .420:31 .15385 .421:34 .00000 .60000 .421:35 .00000 .221:34 .00000 .20000 .421:35 .00000 .221:34 .00000 .20000 .421:35 .00000 .221:34 .00000 .221:34 .00000 .221:34 .00000 .221:34 .00000 .221:34 .00000 .221:34 .00000 .221:41 .00000 <td< td=""><td>A15:22</td><td>.16000</td><td>.04000</td><td>.01000</td><td>A15:22</td><td>.04000</td><td>.02000</td><td>.00250</td><td>.06250</td><td>A15:22</td><td>.64000</td><td>.32000</td><td>.04000</td><td></td></td<>	A15:22	.16000	.04000	.01000	A15:22	.04000	.02000	.00250	.06250	A15:22	.64000	.32000	.04000	
A18:25 .00000 .02000 .03000 A18:25 .00000 .01750 A18:25 .00000 .57143 .42857 A19:30 .02000 .03000 .00000 A19:30 .05000 .01500 .00000 .20000 .419:35 .25000 .5000 .00000 A20:31 .06000 .02000 A20:31 .01500 .00000 .2000 .419:35 .25000 .50000 .00000 A21:32 .08000 .04000 .02131 .01500 .00500 .02131 .5000 .50000 .00000 A22:33 .00000 .04000 .02233 .1588 .0231 .15385 .00000 .40000 .42333 .00000 .40000 .42333 .00000 .40000 .40000 .42633 .00000 .40000 .40000 .42633 .00000 .60000 .40000 .42641 .00000 .21641 .00000 .42641 .00000 .42641 .00000 .42644 .00000 .42644 .00000 .22640	A16:23	.08000	.06000	.02000	A16:23	.02000	.03000	.00500	.05500	A16:23	.36364	.54545	.09091	
A19:30 -02000 .03000 .00000 A19:30 .00500 .00000 .02000 A19:30 .25000 .75000 .00000 A20:31 .05000 .00000 .420:31 .01500 .00000 .20200 .420:31 .5000 .5000 .00000 A21:32 .08000 .08000 .02000 .421:32 .36364 .54545 .09001 A22:33 .04000 .08000 .421:33 .00000 .05500 A21:32 .3586 .65464 .54645 .00000 A22:33 .00000 .06000 A23:34 .00000 .02000 .60000 .420:00 .60000 .40000 .60000 .60000 .60000 .40000 .60000 .	A17:24	.00000	.04000	.04000	A17:24	.00000	.02000	.01000	.03000	A17:24	.00000	.66667	.33333	
A20:31 .06000 .03000 .00000 A20:31 .50000 .50000 .60000 A21:32 .08000 .06000 A20:31 .50000 .50000 .60000 .60000 .421:32 .36364 .56454 .09091 A22:33 .04000 .99000 .04000 A21:32 .20000 .03000 .05000 A21:32 .35364 .56454 .99991 A22:33 .00000 .06000 A22:33 .01000 .06500 A22:33 .05300 .60000 .40000 A24:35 .00000 .02000 .02000 .05000 .01500 .03000 A24:35 .00000 .00000 .40000 .420:35 .00000 .00000 .426:41 .00000 .01000 .02000 .02000 .00000 .00000 .02000 .02000 .02000 .02000 .02000 .02000 .02000 .02000 .02000 .02000 .02000 .02000 .02000 .02000 .02000 .02000 .02000 .02000	A18:25	.00000	.02000	.03000	A18:25	.00000	.01000	.00750	.01750	A18:25	.00000	.57143	.42857	
A21:32 08000 .06000 .02000 A21:32 .02000 .02000 .02000 .02000 .02000 .02000 .02000 .021:32 .36364 .54545 .09091 A22:33 .04000 .08000 .421:33 .01000 .06500 A22:33 .1588 .69231 .1588 A23:34 .00000 .08000 .423:34 .00000 .62000 .422:34 .00000 .60000 .40000 A24:35 .00000 .02000 .02000 .02000 .02000 .02000 .60000 .60000 .420:33 A24:35 .00000 .02000 .02000 .00000 .1000 .00000 .6	A19:30	.02000	.03000	.00000	A19:30	.00500	.01500	.00000	.02000	A19:30	.25000	.75000	.00000	
A22:33 .04000 .09000 .04000 A22:33 .01000 .06500 A22:33 .15385 .69231 .15385 A23:34 .00000 .08000 A23:34 .00000 .05000 A22:33 .0000 .60000 .40000 A24:35 .00000 .03000 .05000 .424:35 .00000 .50000 .50000 .50000 .60000 .40000 A24:35 .00000 .02000 .00000 A24:45 .00000 .50000 .50000 .50000 .50000 .50000 .60000 .60000 .60000 .60000 .60000 .60000 .60000 .60000 .60000 .60000 .60000 .60000 .60000 .60000 .60000 .60000 .60000 .60000 .66667 .33333 .66667 .60000 .62000 .60000 .62000 .62000 .62000 .62000 .62000 .62000 .62000 .62000 .62000 .62000 .62000 .62000 .62000 .62647 .00000 <t< td=""><td>A20:31</td><td>.06000</td><td>.03000</td><td>.00000</td><td>A20:31</td><td>.01500</td><td>.01500</td><td>.00000</td><td>.03000</td><td>A20:31</td><td>.50000</td><td>.50000</td><td>.00000</td><td></td></t<>	A20:31	.06000	.03000	.00000	A20:31	.01500	.01500	.00000	.03000	A20:31	.50000	.50000	.00000	
A23:34 .00000 .06000 .A23:34 .00000 .2000 .2000 .2000 .40000 .40000 .40000 A24:35 .00000 .03000 .60000 .424:35 .00000 .03000 .60000 .40000 .424:35 .00000 .01500 .03000 .425:40 .00000 .425:40 .00000 .425:40 .00000 .425:40 .00000 .426:41	A21:32	.08000	.06000	.02000	A21:32	.02000	.03000	.00500	.05500	A21:32	.36364	.54545	.09091	
A24:35 .00000 .03000 .06000 A24:35 .00000 .01500 .03000 A24:35 .00000 .50000 .50000 .50000 .50000 .50000 .50000 .42:35 .00000 .01500 .01500 .01000 .02000 .00000 .00000 .42:35 .00000 .00000 .42:35 .00000 .00000 .00000 .00000 .00000 .42:41 .00000 .00000 .00000 .42:41 .00000 .42:41 .00000 .42:41 .00000 .42:41 .00000 .42:43 .00000 .00000 .42:43 .00000 .00000 .42:43 .00000 .00000 .42:43 .00000 .00000 .42:43 .00000 .00000 .42:43 .00000 .00000 .42:43 .00000 .00000 .42:43 .00000 .00000 .42:43 .00000 .00000 .42:43 .00000 .00000 .42:43 .00000 .00000 .42:43 .00000 .00000 .42:43 .00000 .00000 .42:43<	A22:33	.04000	.09000	.04000	A22:33	.01000	.04500	.01000	.06500	A22:33	.15385	.69231	.15385	
A25:40 .00000 .02000 .00000 A25:40 .00000 .02000 .00000 .02000 .000000 .00000 .00000	A23:34	.00000	.06000	.08000	A23:34	.00000	.03000	.02000	.05000	A23:34	.00000	.60000	.40000	
A26:41 .00000 .02000 .00000 A26:41 .00000 .02000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .02000 .03000 .02000 .03000 .02100 .03010 .00000 .02500 .00000 .02500 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000<	A24:35	.00000	.03000	.06000	A24:35	.00000	.01500	.01500	.03000	A24:35	.00000	.50000	.50000	
A27:42 .00000 .04000 .04000 A27:42 .00000 .02000 .01000 .03000 A27:42 .00000 .66667 .33333 A28:43 .00000 .06000 .08000 .03000 .02000 .65000 .428:43 .00000 .60000 .40000 .428:43 .00000 .05000 .428:43 .00000 .428:43 .00000 .60000 .40000 .428:43 .00000 .05000 .429:44 .00000 .33333 .66667 A30:45 .00000 .02000 .03000 .03000 .04000 .429:44 .00000 .33333 .66667 A31:50 .00000 .00500 .00500 .00500 .00500 .00500 .00000 .00000 .00000 .431:50 .00000 .00500 .431:50 .00000 .431:50 .00000 .00500 .431:50 .00000 .00000 .431:50 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 .00000 <td>A25:40</td> <td>.00000</td> <td>.02000</td> <td>.00000</td> <td>A25:40</td> <td>.00000</td> <td>.01000</td> <td>.00000</td> <td>.01000</td> <td>A25:40</td> <td>.00000</td> <td>1.0000</td> <td>.00000</td> <td></td>	A25:40	.00000	.02000	.00000	A25:40	.00000	.01000	.00000	.01000	A25:40	.00000	1.0000	.00000	
A28:43 .00000 .08000 A28:43 .00000 .02000 .02000 .02000 .428:43 .00000 .40000 A28:44 .00000 .04000 .04000 .06000 .429:44 .00000 .3333 .66667 A30:45 .00000 .02000 .04000 .03000 .04000 .3333 .66667 A30:45 .00000 .02000 .04000 .03000 .04000 .33333 .66667 A30:45 .00000 .01000 .03000 .04000 .33045 .00000 .25000 .75000 A31:50 .00000 .01000 .00000 .00500 .00000 </td <td>A26:41</td> <td>.00000</td> <td>.02000</td> <td>.00000</td> <td>A26:41</td> <td>.00000</td> <td>.01000</td> <td>.00000</td> <td>.01000</td> <td>A26:41</td> <td>.00000</td> <td>1.0000</td> <td>.00000</td> <td></td>	A26:41	.00000	.02000	.00000	A26:41	.00000	.01000	.00000	.01000	A26:41	.00000	1.0000	.00000	
A29:44 .00000 .04000 .16000 A29:44 .00000 .02000 .12000 .04000 .04000 .04000 .04000 .04000 .03333 .6667 A30:45 .00000 .02000 .12000 .03000 .03000 .04000 .03303 .6667 A31:50 .00000 .01000 .03000 .04000 .431:50 .00000 .03000 .02000 .02000 .03000 .03000 .00000	A27:42	.00000	.04000	.04000	A27:42	.00000	.02000	.01000	.03000	A27:42	.00000	.66667	.33333	
A30:45 .00000 .02000 .12000 A30:45 .00000 .03000 .04000 A30:45 .00000 .75000 A31:50 .00000 .01000 .00000 .00500 .00000 .431:50 .00000 .25000 .75000 A31:50 .00000 .00000 .00500 .00000 .00500 .00000 .00000 .00000 A32:51 .00000 .02000 .03000 .03000 .00000 <	A28:43	.00000	.06000	.08000	A28:43	.00000	.03000	.02000	.05000	A28:43	.00000	.60000	.40000	
A31:50 .00000 .01000 .00000 A31:50 .00000 .00500 A31:50 .00000 .00000 .00000 .00000 .00000 A31:50 .00000 .00000 A31:50 .00000 .00000 .00500 .00500 .00500 A31:50 .00000 .50000<	A29:44	.00000	.04000	.16000	A29:44	.00000	.02000	.04000	.06000	A29:44	.00000	.33333	.66667	
A32:51 .00000 .01000 .00000 A32:51 .00000 .00500 .00500 .00500 .00000<	A30:45	.00000	.02000	.12000	A30:45	.00000	.01000	.03000	.04000	A30:45	.00000	.25000	.75000	
A33:52 .00000 .03000 A33:52 .00000 .01750 A33:52 .00000 .57143 .42857 A34:53 .00000 .03000 .03000 .03150 .01500 .01750 A33:52 .00000 .57143 .42857 A34:53 .00000 .02000 .03000 .01500 .01500 .03000 .50000 .50000 A36:55 .00000 .02000 .01000 .03000 .04000 A35:54 .00000 .50000 A36:55 .00000 .01000 .00000 .022750 A36:55 .00000 .18182 .81818 Bm .25000 .50000 .50000 Bm .25000 .50000 .85000	A31:50	.00000	.01000	.00000	A31:50	.00000	.00500	.00000	.00500	A31:50	.00000	1.0000	.00000	
A34:53 .00000 .03000 .06000 A34:53 .00000 .01500 .03000 A34:53 .00000 .50000 .50000 A35:54 .00000 .02000 .12000 A35:54 .00000 .03000 .04000 A35:54 .00000 .75000 A36:55 .00000 .01000 .00500 .02250 .02750 A36:55 .00000 .18182 .81818 Bm .25000 .50000 .25000 .50000 .25000 .50000 .8000 .50000	A32:51	.00000	.01000	.00000	A32:51	.00000	.00500	.00000	.00500	A32:51	.00000	1.0000	.00000	
A35:54 .00000 .02000 .12000 A35:54 .00000 .03000 .04000 A35:54 .00000 .75000 A36:55 .00000 .01000 .00500 .02250 .02750 A36:55 .00000 .18182 .81818 Bm .25000 .50000 .50000 .50000 .50000 .50000 .25000 .50000 .25000 .50000 .25000 .50000 .25000 .50000 .25000 .50000 .25000 .50000 .25000 .50000 .25000 .50000 .25000 .50000 .25000 .50000 .25000 .50000 .25000 .50000 .25000 .50000 .25000 .50000 .25000 .50000 .25000 .50000 .25000 .50000 .25000 .25000 .50000 .25000 .25000 .25000 .25000 .25000 .25000 .25000 .25000 .25000 .25000 .25000 .25000 .25000 .25000 .25000 .25000 .25000 .25000 <	A33:52	.00000	.02000	.03000	A33:52	.00000	.01000	.00750	.01750	A33:52	.00000	.57143	.42857	
A36:55 .00000 .01000 .09000 A36:55 .00000 .02500 .02750 A36:55 .00000 .18182 .81818 Bm .25000 .50000 .25000	A34:53	.00000	.03000	.06000	A34:53	.00000	.01500	.01500	.03000	A34:53	.00000	.50000	.50000	
Bm .25000 .50000 .25000 Bm .25000 .50000 .25000 1.0000 Bm .25000 .50000 .25000	A35:54	.00000	.02000	.12000	A35:54	.00000	.01000	.03000	.04000	A35:54	.00000	.25000	.75000	
	A36:55	.00000	.01000	.09000	A36:55	.00000	.00500	.02250	.02750	A36:55	.00000	.18182	.81818	
9	Bm	.25000	.50000	.25000	Bm	.25000	.50000	.25000	1.0000	Bm	.25000	.50000	.25000	-
														9

Multiple Observations in General

Conditional Probability

$$\Pr\left(A_{i1}, A_{i2}, \dots, A_{in} | B_j\right) = \prod_{k=1}^n \Pr\left(A_{ik} | B_j\right)$$

Posterior Probability

$$\Pr(B_j|A_{i1}, A_{i2}, \dots, A_{in}) = \frac{\Pr(A_{i1}, A_{i2}, \dots, A_{in}|B_j)\Pr(B_j)}{\sum_{j=1}^b \Pr(A_{i1}, A_{i2}, \dots, A_{in}|B_j)\Pr(B_j)}$$

$$\Pr(B_j|A_{i1}, A_{i2}, \dots, A_{in}) = \frac{\left(\prod_{k=1}^n \Pr(A_{ik}|B_j)\right)\Pr(B_j)}{\sum_{j=1}^b \left(\left(\prod_{k=1}^n \Pr(A_{ik}|B_j)\right)\Pr(B_j)\right)}$$

$$\propto \left(\prod_{k=1}^n \Pr(A_{ik}|B_j)\right)\Pr(B_j)$$

Two Test Scores

Note that, with this example (CPT) the sum of two test scores is not sufficient. For example the posterior probability of B given sum = 6 are shown below.

Conditional Probability of A	Joint Probability of A and B	
given B (Model)	and the Marginal	Posterior Probability
PROB1 B1:L B2:M B3:H	PROB2 B1:L B2:M B3:H	Am PROB3 B1:L B2:M B3:H
A12:15 .00000 .01000 .00000	A12:15 .00000 .00500 .00000 .00	A12:15 .00000 1.0000 .00000
A32:51 .00000 .01000 .00000	A32:51 .00000 .00500 .00000 .00	A32:51 .00000 1.0000 .00000
A17:24 .00000 .04000 .04000	A17:24 .00000 .02000 .01000 .03	A17:24 .00000 .66667 .33333
A27:42 .00000 .04000 .04000	A27:42 .00000 .02000 .01000 .03	A27:42 .00000 .66667 .33333
A22:33 .04000 .09000 .04000	A22:33 .01000 .04500 .01000 .06	500 A22:33 .15385 .69231 .15385

If CPT is proportional to the PDF of the Normal distribution, sum is the sufficient statistics.

Numerical Category Values

When the categories have numerical value, CPT can be expressed by a formula f(.|.).

Let A and B be the random variables whose values are α_i and β_j .

CPT

11

$$\Pr(A_i|B_j) = \Pr(A = \alpha_i|B = \beta_j) = f(\alpha_i|\beta_j)$$

Coin Toss: Bernoulli Distribution

A: Indicator Variable

$$A = \begin{cases} 1 & \text{Head} \\ 0 & \text{Tail} \end{cases} \quad \alpha = \begin{cases} 1 & \text{Head} \\ 0 & \text{Tail} \end{cases}$$

B: Success Probability

$$0 \le \beta_j \le 1$$

 $\Pr(A = 1|B = \beta_j) = \beta_j \quad \& \quad \Pr(A = 0|B = \beta_j) = 1 - \beta_j$

CPT: Bernoulli Distribution

$$\Pr\left(A = \alpha | B = \beta_j\right) = f(\alpha | \beta_i) = \beta_i^{\alpha} (1 - \beta_j)^{1 - \alpha}$$

13

Coin Toss: Binomial Distribution

Let Y be the random number representing the number of Heads out or N tosses. There are $\begin{pmatrix} N \\ y \end{pmatrix}$ sequences of $A_{(k)}, k = 1, 2, ..., N$, which result in y Heads, i.e.

$$\sum_{k=1}^N \alpha_{(k)} = y$$

CPT: Binomial Distribution

$$\Pr(Y = y | B_j = \beta_j) = \text{number of combinations} \times \\ \Pr\left(\sum_{k=1}^N \alpha_{(k)} = y | B_j = \beta_j\right) \\ = f(y | \beta_j) \\ = \binom{N}{y} \times \beta_j^y (1 - \beta_j)^{N-y}$$

Note that when N=1, Binomial = Bernoulli.

Coin Toss: Binomial Distribution

Two observations: $A_{(1)}$ and $A_{(2)}$

$$\begin{aligned} \Pr\left(A_{(1)} = \alpha_{(1)} \& A_{(2)} = \alpha_{(2)} | B_j = \beta_j\right) &= \Pr\left(A_{(1)} = \alpha_{(1)} | B_j = \beta_j\right) \times \Pr\left(A_{(2)} = \alpha_{(2)} | B_j = \beta_j\right) \\ &= f(\alpha_{(1)} | \beta_j) \times f(\alpha_{(2)} | \beta_j) \\ &= \beta_j^{\alpha_{(1)}} (1 - \beta_j)^{1 - \alpha_{(1)}} \times \beta_j^{\alpha_{(2)}} (1 - \beta_j)^{1 - \alpha_{(2)}} \\ &= \beta_j^{\alpha_{(1)} + \alpha_{(2)}} (1 - \beta_j)^{2 - (\alpha_{(1)} + \alpha_{(2)})} \end{aligned}$$

N observations: $A_{(k)}, k = 1, 2, \ldots, N$

$$\Pr\left(A_{(1)} = \alpha_{(1)}\&A_{(2)} = \alpha_{(2)}, \dots, A_{(N)} = \alpha_{(N)}|B_j = \beta_j\right) \\ = \prod_{k=1}^N f(A_{(k)} = \alpha_{(k)}|\beta_j) \\ = \prod_{k=1}^N \beta_j^{\alpha_{(k)}} (1 - \beta_j)^{1 - \alpha_{(k)}} \\ = \beta_j^{\sum_{k=1}^N \alpha_{(k)}} (1 - \beta_j)^{N - \sum_{k=1}^N \alpha_{(k)}} \\ \propto \Pr\left(\text{ number of Heads out of } N = \sum_{k=1}^N \alpha_{(k)}|B_j = \beta_j\right)$$

14

Binomial Distribution

CPT:

$$\Pr\left(Y = y | B = \beta_j\right) = {N \choose y} \times \beta_j^y (1 - \beta_j)^{N-y}$$
$$Y = \{0, 1, 2, \dots, N\}$$

Joint Probability

$$\Pr(Y = y, B = \beta_j) = \Pr(Y = y|B = \beta_j) \times \Pr(B = \beta_j)$$
$$= \binom{N}{y} \times \beta_j^y (1 - \beta_j)^{N-y} \Pr(B = \beta_j)$$

Posterior Probability

$$\Pr\left(B = \beta_j | Y = y\right) \quad \propto \quad \Pr\left(Y = y, B = \beta_j\right) \; \propto \; \beta_j^y (1 - \beta_j)^{N-y} \Pr\left(B = \beta_j\right)$$

$$\Pr(B = \beta_j | Y = y) = \frac{\Pr(Y = y, B = \beta_j)}{\sum_{j=1}^b \Pr(Y = y, B = \beta_j)}$$
$$= \frac{\beta_j^y (1 - \beta_j)^{N-y} \Pr(B = \beta_j)}{\sum_{j=1}^b \beta_j^y (1 - \beta_j)^{N-y} \Pr(B = \beta_j)}$$

Extending B to continuous RV

About continuous random variables

Discrete Random Variables

probability function

 $f(x) = \Pr(X = x) \qquad \qquad x_0 \le X \le x_1$ $\sum_{i=1}^{x_1} f(x) = 1$

distribution function

 $x = x_0$

$$\Pr\left(X \le x\right) = F(x) = \sum_{z=x_0}^{x} f(z)$$

$$\Pr(a < X \le b) = \sum_{x=x_0}^{b} f(x) - \sum_{x=x_0}^{a} f(x) = F(b) - F(a)$$
$$F^{-1}(\alpha) = \min x \in R : p \le F(x)$$

Continuous Random Variables

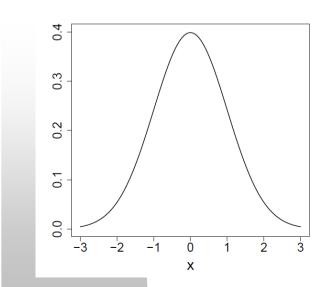
probability distribution function and density (pdf)

$$\Pr\left(X \le x\right) = F(x) = \int_{-\infty}^{x} f(z)dz \qquad \qquad \int_{X} f(x)dx = 1$$

probabilities

$$\Pr (a \le X \le b) = \int_{a}^{b} f(x) dx$$
$$\Pr (X = x) = 0$$
$$\Pr (x - \varepsilon < X < x + \varepsilon) \propto f(x)$$

pdf



17

Joint PDF, Conditional PDF

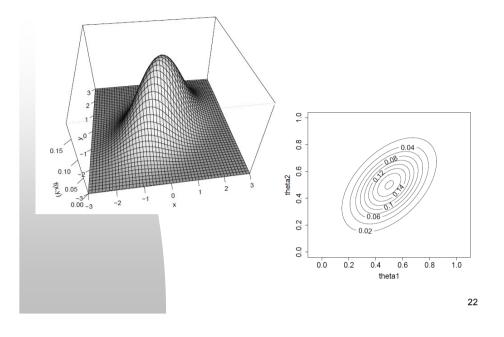
joint pdf

$$\int_Y \int_X f(x,y) dx dy = 1$$

conditional pdf

$$\begin{split} f(x|y) &= \frac{f(x,y)}{f(y)} \qquad f(y|x) = \frac{f(x,y)}{f(x)} \\ \int_X f(x|y) dx &= 1, \qquad \int_Y f(y|x) dy = 1 \end{split}$$

Joint PDF



21	

		B is discrete	B is discrete		
		categorical	numerical	numerical	
A is discrete	categorical	A	D	G	
	numerical	В	E cpt=pf,	H h=pf cpt=pf, h=	
A is continuous	numerical	С	F cpt=pdf	I .h=of cot=odf.h=	

A: simplest case B: test score C: body hight M/F

D: E: coin toss F/NF F: battery test

G: H: Bernoulli/Binomial I: Normal

Bayes Theorem (continuous)

Bayes theorem

$$h(\theta|y) = \frac{f(\theta, y)}{f(y)} = \frac{f(y|\theta)h(\theta)}{\int_{\Theta} f(y|\theta)h(\theta)d\theta}$$

proportionality

$$egin{aligned} h(heta|y) &=& rac{1}{\int_{\Theta}f(y| heta)h(heta)d heta}f(y| heta)h(heta)\ &\propto& f(y| heta)h(heta) \end{aligned}$$

 Posterior pdf is proportional to Likelihood x Prior pdf

Bayesian Inference

Specify the model distribution (likelihood.)

 $f(y|\theta)$

 θ is called the parameter of the model.

25

27

 Express your prior belief on the model parameter as the prior distribution. (prior pdf)

h(heta)

 Update your prior belief on the model parameter by combining the prior pdf and the likelihood.

$$\begin{array}{lll} h(\theta|y) &=& \displaystyle \frac{1}{\int_{\Theta} f(y|\theta) h(\theta) d\theta} f(y|\theta) h(\theta) \\ &\propto & \displaystyle f(y|\theta) h(\theta) \end{array}$$

Non-informative Prior

 \blacksquare If we do not have any knowledge on θ

 $h(\theta) = 1$

• Posterior distribution of θ

$$\begin{split} h(\theta|y) &\propto f(y|\theta)h(\theta) = f(y|\theta) \\ &= \theta^y (1-\theta)^{N-y} \\ \int_0^1 \theta^y (1-\theta)^{N-y} d\theta = B(y+1,N-y+1) \\ h(\theta|y) &= \frac{1}{B(y+1,N-y+1)} \theta^y (1-\theta)^{N-y} \\ \text{ of Beta distribution} \\ B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \end{split}$$

Going back to Binomial Likelihood

Based on the # of successes out of N trials, we must make inference on the parameter q which is the probability of success.

$$eta_j, \ j = 1, 2, \dots, b, \quad b \to +\infty$$
 $eta_i = heta$

How do we specify the prior distribution?

Beta Distribution

• pdf: X ~ Beta(α , β)

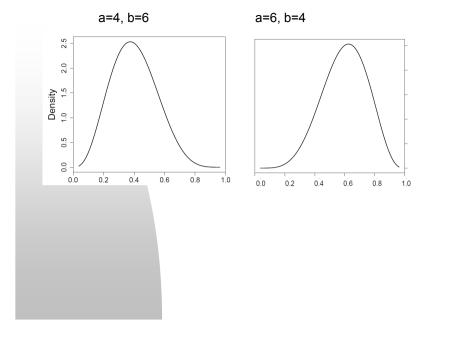
$$f(x|\alpha,\beta) = \frac{1}{B(\alpha,\beta)} \times x^{\alpha-1}(1-x)^{\beta-1}$$

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

moments

$$\mathcal{E}(X) = \frac{\alpha}{\alpha + \beta}, \quad var(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$
$$mode(X) = \frac{\alpha - 1}{\alpha + \beta - 2} (\alpha > 1\&\beta > 1)$$

Beta Distribution



Summary of Posterior Distribution

Posterior Summary

$$\begin{aligned} \boldsymbol{\mathcal{E}}(\boldsymbol{\theta}|\boldsymbol{y}) &= \frac{y+1}{N+2} \\ mode(\boldsymbol{\theta}|\boldsymbol{y}) &= \frac{y}{N} \\ var\left(\boldsymbol{\theta}|\boldsymbol{y}\right) &= \frac{(y+1)(N-y+1)}{(N+2)^2(N+3)} \end{aligned}$$

Identifying Posterior Distribution

If we know

$$h(\theta|y) \propto \theta^{y+\alpha_0-1} (1-\theta)^{N-y+\beta_0-1}$$

the we know

$$\theta \mid \sim \text{Beta}(y + \alpha_0, N - y + \beta_0)$$

30

How to Choose Point Estimate

Expected loss Choose d which minimizes:

$$\mathcal{E}L(d|oldsymbol{y})=\mathcal{E}_{\Theta}(L(d, heta)|oldsymbol{y}) \hspace{1.5cm} = \hspace{1.5cm} \int_{\Theta}L(d, heta)h(heta|oldsymbol{y})d heta$$

Various types of expected loss

minimized by:

Squared Error Loss	=	$(d- heta)^2$	mean
Bilinear Error Loss	=	$\left\{ egin{array}{ll} a(d- heta) & ext{if} \ d\leq heta\ b(heta-d) & ext{if} \ d> heta \end{array} ight.$	median
0-1 Error Loss	=	$\left\{ egin{array}{ll} 0 & ext{if} \; d- heta \leq b \ 1 & ext{if} \; d- heta > b \end{array} ight.$	mode

Point Estimate

Squared error loss

$$\begin{aligned} \mathcal{E}L(d|\boldsymbol{y}) &= \int_{\Theta} (d-\theta)^2 h(\theta|\boldsymbol{y}) d\theta &= \int_{\Theta} \left(d^2 - 2d\theta + \theta^2 \right) h(\theta|\boldsymbol{y}) d\theta \\ &= d^2 - 2d \int_{\Theta} \theta h(\theta|\boldsymbol{y}) d\theta + \int_{\Theta} \theta^2 h(\theta|\boldsymbol{y}) d\theta \end{aligned}$$

Minimum by differentiation

$$d = \int_{\Theta} heta h(heta | oldsymbol{y}) d heta = oldsymbol{\mathcal{E}}(heta | oldsymbol{y})$$

Summary of Posterior Distribution

Posterior summary in general

$$\boldsymbol{\mathcal{E}}(\boldsymbol{\theta}|\boldsymbol{y}) = \frac{\alpha_0 + y}{\alpha_0 + \beta_0 + N},$$
$$mode(\boldsymbol{\theta}|\boldsymbol{y}) = \frac{\alpha_0 + y - 1}{\alpha_0 + \beta_0 + N - 2}$$
$$var(\boldsymbol{\theta}|\boldsymbol{y}) = \frac{(\alpha_0 + y)\beta_0 + N - y}{(\alpha_0 + \beta_0 + N)^2(\alpha_0 + \beta_0 + N + 1)}$$

Natural Conjugate Prior

If you can express you prior knowledge of θ by a Beta distribution:

$$h(\theta) = \frac{1}{B(\alpha_0, \beta_0)} \theta^{\alpha_0 - 1} (1 - \theta)^{\beta_0 - 1}$$

$$\propto \theta^{\alpha_0 - 1} (1 - \theta)^{\beta_0 - 1}$$

Posterior is also a Beta distribution.

$$h(\theta|y) \propto \theta^{y}(1-\theta)^{N-y} \times \theta^{\alpha_{0}-1}(1-\theta)^{\beta_{0}-1}$$

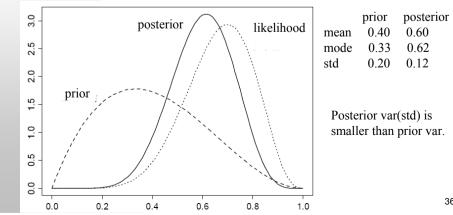
= $\theta^{y+\alpha_{0}-1}(1-\theta)^{N-y+\beta_{0}-1}$
= pdf of $Beta(y+\alpha_{0}, N-y+\beta_{0})$

34

36

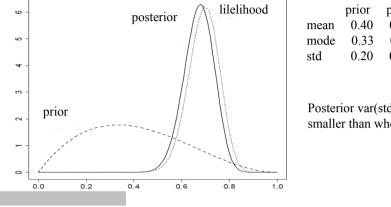
Example

- Ten tosses of a coin: HTHHTHHHHT N=10, y=7
- Prior: B(2,3)
- Posterior: B(9,6) = B(2+7, 3+10-7)



Example

- 50 tosses of a coin: N=50, y=35
- Prior: B(2,3)
- Posterior: B(37,18)



posterior 0.67 0.68 0.06

Posterior var(std) is smaller than when N=10.

37

Posterior Variance

Expected to be small

$$\boldsymbol{\mathcal{E}}(\theta|y) = var\left(\theta\right) - var\left(\boldsymbol{\mathcal{E}}(\theta|y)\right)$$

Some exceptions Beta(10,2) prior: data: n=3, x=1 posterior: Beta(11,4)

Beta(10,2) n=10, x=1 Beta(11,11)

Predictive Distribution

posterior predictive What do we know about the next observation after observing N obs.

$$h(\tilde{y}|\boldsymbol{y}) = \int f(\tilde{y}|\pi)h(\pi|\boldsymbol{y})d\pi$$

= $\int \pi^{\tilde{y}}(1-\pi)^{1-\tilde{y}} \times \frac{1}{B(\alpha_{1},\beta_{1})}\pi^{\alpha_{1}-1}(1-\pi)^{\beta_{1}-1}$
= $\frac{1}{B(\alpha_{1},\beta_{1})}\int \pi^{\alpha_{1}+\tilde{y}-1}(1-\pi)^{\beta_{1}-\tilde{y}}$
= $\frac{B(\alpha_{1}+\tilde{y},\beta_{1}+1-\tilde{y})}{B(\alpha_{1},\beta_{1})}$
= $\frac{\Gamma(\alpha_{1}+\tilde{y})\Gamma(\beta_{1}+1-\tilde{y})}{\Gamma(\alpha_{1}+\beta_{1}+1)} \times \frac{\Gamma(\alpha_{1}+\beta_{1})}{\Gamma(\alpha_{1})\Gamma(\beta_{1})}$
= $\frac{1}{\alpha_{1}+\beta_{1}} \times \frac{(\alpha_{1}+\tilde{y}-1)!(\beta_{1}-\tilde{y})!}{(\alpha_{1}-1)!(\beta_{1}-1)!}$

Normal Model

voltage of battery

α	$\beta = 1.2$	$\beta = 1.$	5
0.95	0.001	0.000	
1.00	0.005	0.000	
1.05	0.030	0.000	
1.10	0.104	0.000	
1.15	0.220	0.000	
1.20	0.282	0.000	
1.25	0.220	0.001	
1.30	0.104	0.005	
1.35	0.030	0.030	
1.40	0.005	0.104	
1.45	0.001	0.220	
1.50	0.000	0.282	
1.55	0.000	0.220	
1.60	0.000	0.104	
1.65	0.000	0.030	
1.70	0.000	0.005	
1.75	0.000	0.001	
$\Pr(B)$	$=\beta_j A $	$= \alpha_i)$	=

A digital voltage meter is used to measure the voltage of a battery whose true voltage is either 1.2v or 1.5v.

The digital voltage meter display the voltage with the step of 0.05 volt as shown in the CPT.

$$B = \beta_j | A = \alpha_i) = \frac{\frac{1}{c_j} \exp(-\frac{1}{2}(\alpha_i - \beta_j)^2 / 0.05) \operatorname{Pr}(B = \beta_j)}{\sum_k^a \frac{1}{c_k} \exp(-\frac{1}{2}(\alpha_i - \beta_k)^2 / 0.05) \operatorname{Pr}(B = \beta_k)}$$