

Introduction to Bayesian Statistics

Shin-ichi Mayekawa

mayekawa@hum.titech.ac.jp
www.ms.hum.titech.ac.jp

Where to find the handout

www.ms.hum.titech.ac.jp



2

Outline

1. Introduction to Bayes Theorem
2. Binomial Model and Poisson Model
3. Normal Model
4. Normal Regression Model
5. Bayesian Numerical Calculations
(Markov Chain Monte Carlo)
6. Application of MCMC
7. Model Selection
8. miscellaneous topics

3

Example: ST or SW

- We know that:
 - 90% of the ST fans likes Tea,
 - 30% of the SW fans likes Tea.And
 - the ratio of ST fans vs SW fans = 1:9 .
- What is the probability that a person is a ST fan if we find out that he/she likes Tea?

4

ST or SW (mathematical formulation)

- Two-way classification
A: A1=Tea or A2=Coffee (not Tea) .
B: B1=ST or B2=SW
- Model Probability (Likelihood, CPT):
90% of the ST fans likes Tea,
 $\Pr(\text{Tea} | \text{ST}) = 0.9$
30% of the SW fans likes Tea
 $\Pr(\text{Tea} | \text{SW}) = 0.3$

5

ST or SW (mathematical formulation)

- Prior or Marginal Probability
The ratio of ST fans vs SW fans = 1:9
 $\Pr(\text{ST}) = 0.1$ and $\Pr(\text{SW}) = 0.9$
- Posterior Probability
(inverse probability)
What is the probability that a person is a ST fan if we know that he/she likes Tea?
 $\Pr(\text{ST} | \text{Tea}) \leftarrow \Pr(\text{Tea} | \text{ST}) \text{ etc}$

6

ST or SW (solution)

- Given the model probability and the prior probability, we must find the posterior probability.
- We know:
 $\Pr(\text{Tea} | \text{ST})$ and $\Pr(\text{Tea} | \text{SW})$
 $\Pr(\text{ST})$ and $\Pr(\text{SW})$
- Using Bayes Theorem, we can calculate the posterior probability as
 $\Pr(\text{ST} | \text{Tea}) \propto \Pr(\text{Tea} | \text{ST}) \times \Pr(\text{ST})$

7

ST or SW (joint frequency table)

- Suppose that we know the Joint Frequency

		Cross Table		Marginal Freq
		sw	st	
		B1	B2	marginal
tea	A1	270	90	360
coffe	A2	630	10	640
marginal		900	100	1000 total

- Then, it is easy to calculate the desired Posterior Probability as the row conditionals.

$$\begin{aligned} \Pr(\text{ST} | \text{Tea}) &= 90 / 360 = 0.25 \\ \Pr(\text{SW} | \text{Tea}) &= 270 / 360 = 0.75 \\ \Pr(\text{ST} | \text{Coffee}) &= 10 / 640 = 0.016 \\ \Pr(\text{SW} | \text{Coffee}) &= 630 / 640 = 0.984 \end{aligned}$$

8

ST or SW (joint probability table)

- Joint Probability Table
joint prob = joint freq / total

		Joint Probability P(A,B)		Marginal Probability P(A)
		sw	st	
tea coffe	A1	0.27	0.09	0.36
	A2	0.63	0.01	0.64
	P(B)	0.9	0.1	1

- Posterior Probability as the row conditional
Posterior Probability P(B|A)

		sw		st	sum
		B1	B2		
tea	A1	0.750	0.250		1
coffe	A2	0.984	0.016		1

9

ST or SW (how to calculate joint)

- Joint Probability $\Pr(A_i, B_j)$ from Model Prob $\Pr(A_i | B_j)$ and Prior $\Pr(B_j)$

$$\Pr(A_i, B_j) = \Pr(A_i | B_j) \times \Pr(B_j)$$

$$\begin{aligned} \Pr(\text{Tea}, \text{ST}) &= \Pr(\text{Tea} | \text{ST}) \times \Pr(\text{ST}) \\ \Pr(\text{Tea}, \text{SW}) &= \Pr(\text{Tea} | \text{SW}) \times \Pr(\text{SW}) \\ \Pr(\text{Coffee}, \text{ST}) &= \Pr(\text{Coffee} | \text{ST}) \times \Pr(\text{ST}) \\ \Pr(\text{Coffee}, \text{SW}) &= \Pr(\text{Coffee} | \text{SW}) \times \Pr(\text{SW}) \end{aligned}$$

- Marginal Probability $\Pr(A_i)$
 $\Pr(A_i) = \text{sum over } B_j \text{ of } \Pr(A_i, B_j)$
- Posterior Probability $\Pr(B_j | A_i)$
 $\Pr(B_j | A_i) = \Pr(A_i, B_j) / \Pr(A_i)$

10

ST or SW (joint probability table)

		Conditional Probability P(A B)		
		sw	st	
tea coffe	A1	0.3	0.9	tea
	A2	0.7	0.1	coffe
	sum	1	1	

		Joint Probability P(A,B)		Marginal P
		sw	st	
tea coffe	A1	0.27	0.09	0.36
	A2	0.63	0.01	0.64
	P(B)	0.9	0.1	1

		Prior Probability of B (P(B))		
		sw	st	
tea coffe	B1	0.9	0.1	tea
	B2			coffe

		Posterior Probability P(B A)		sum
		sw	st	
tea coffe	A1	0.750	0.250	1
	A2	0.984	0.016	1

11

Bayes Theorem

Bayes Theorem

Let $A_i, i = 1, 2, \dots, a$, and $B_j, j = 1, 2, \dots, b$ be the two-way classification of the result of an experiment and assume that A_i and B_j are mutually exclusive and exhaustive. Let $\Pr(A_i | B_j)$ be the conditional probability of A_i given B_j , and $\Pr(B_j)$ be the marginal probability of B_j .

Then, the conditional probability of B_j given A_i can be given as

$$\Pr(B_j | A_i) = \frac{\Pr(A_i | B_j) \Pr(B_j)}{\sum_{j=1}^b \Pr(A_i | B_j) \Pr(B_j)} \quad (1)$$

12

Conditional and Joint Prob

Conditional Probability

$$\sum_{i=1}^a \Pr(A_i|B_j) = 1 \quad \text{and} \quad \sum_{j=1}^b \Pr(B_j|A_i) = 1 \quad (2)$$

Joint Probability and the Conditional Probability

$$\Pr(A_i, B_j) = \Pr(A_i|B_j)\Pr(B_j) \quad (3)$$

$$= \Pr(B_j|A_i)\Pr(A_i) \quad (4)$$

$$\sum_{i=1}^a \sum_{j=1}^b \Pr(A_i, B_j) = 1 \quad (5)$$

13

Marginal Probability

Marginal Probability

$$\Pr(A_i) = \sum_{j=1}^b \Pr(A_i, B_j) \quad (6)$$

$$= \sum_{j=1}^b \Pr(A_i|B_j)\Pr(B_j) \quad (7)$$

$$\sum_{i=1}^a \Pr(A_i) = 1 \quad (8)$$

14

Proof of Bayes Theorem

Bayes Theorem

$$\Pr(B_j|A_i) = \frac{\Pr(A_i|B_j)\Pr(B_j)}{\sum_{j=1}^b \Pr(A_i|B_j)\Pr(B_j)} \quad (9)$$

proof

By Eq.(4), we have

$$\Pr(B_j|A_i) = \frac{\Pr(A_i, B_j)}{\Pr(A_i)} \quad (10)$$

However, by Eq.(3), $\Pr(A_i, B_j) = \Pr(A_i|B_j)\Pr(B_j)$ and by Eq.(7), $\Pr(A_i) = \sum_{j=1}^b \Pr(A_i|B_j)\Pr(B_j)$

15

Bayes Theorem (alt)

Bayes Theorem

$$\Pr(B_j|A_i) = \frac{\Pr(A_i|B_j)\Pr(B_j)}{\Pr(A_i)} \quad (11)$$

$$= \frac{\Pr(A_i, B_j)}{\Pr(A_i)} \quad (12)$$

$$\propto \Pr(A_i, B_j) = \Pr(A_i|B_j)\Pr(B_j) \quad (13)$$

Left-hand side is the conditional probability of B_j given A_i .

Therefore, it must satisfy

$$\sum_{j=1}^b \Pr(B_j|A_i) = 1 \quad (14)$$

Therefore

$$\Pr(B_j|A_i) = \frac{1}{\sum_{j=1}^b \Pr(A_i, B_j)} \Pr(A_i, B_j) \quad (15)$$

16

Normalization

Given the Joint probability $\Pr(A_i, B_j)$, the conditional probability $\Pr(B_j|A_i)$ can be calculated by normalizing the joint probability.

17

Interpretation of Bayes Theorem

Cause and Effect Interpretation of Bayes Theorem

Let $A_i, i = 1, 2, \dots, a$, be the possible effects, and $B_j, j = 1, 2, \dots, b$, be the possible causes. That is, the result of an experiment can be interpreted as the effect \times cause two-way classification.

Let $\Pr(A_i|B_j)$ be the conditional probability of A_i given B_j , and $\Pr(B_j)$ be the prior probability of B_j .

Then, the Posterior (or inverse) probability of B_j given A_i can be given as

$$\Pr(B_j|A_i) = \frac{\Pr(A_i|B_j)\Pr(B_j)}{\sum_{j=1}^b \Pr(A_i|B_j)\Pr(B_j)} \quad (16)$$

$\Pr(A_i|B_j)$ is called the Model probability or the Likelihood.

18

Posterior Probability

- The Joint probability is the product of the Model probability (CPT or Likelihood) and the Prior probability.
- The Posterior probability is proportional to the Joint probability.
- Therefore, the Posterior probability is proportional to the product of the Model probability (CPT or Likelihood) and the Prior probability.

19

Conditional Probability Table

CPT:

Contitional Probability Table (Table of $\Pr(A_i|B_j)$) and $\Pr(B_j)$

	B_1	B_2	...	B_j	...	B_3
A_1	$\Pr(A_1 B_1)$	$\Pr(A_1 B_2)$...	$\Pr(A_1 B_j)$...	$\Pr(A_1 B_b)$
A_2	$\Pr(A_2 B_1)$	$\Pr(A_2 B_2)$...	$\Pr(A_2 B_j)$...	$\Pr(A_2 B_b)$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
A_i	$\Pr(A_i B_1)$	$\Pr(A_i B_2)$...	$\Pr(A_i B_j)$...	$\Pr(A_i B_b)$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
A_a	$\Pr(A_a B_1)$	$\Pr(A_a B_2)$...	$\Pr(A_a B_j)$...	$\Pr(A_a B_b)$
$\Pr(B)$	$\Pr(B_1)$	$\Pr(B_2)$...	$\Pr(B_j)$...	$\Pr(B_b)$

When $A_i, i = 1, 2, \dots, a$, are mutually exclusive and exhaustive,

$$\sum_{i=1}^a \Pr(A_i|B_j) = 1$$

That is, the culumn sum of CPT = 1.

20

Joint Probability

Table of the Joint Probability $\Pr(A_i, B_j)$ and the Marginal Probability $\Pr(A_i)$

	B_1	...	B_j	...	B_b	$\Pr(A)$
A_1	$\Pr(A_1 B_1)\Pr(B_1)$...	$\Pr(A_1 B_j)\Pr(B_j)$...	$\Pr(A_1 B_b)\Pr(B_b)$	$\Pr(A_1)$
A_2	$\Pr(A_2 B_1)\Pr(B_1)$...	$\Pr(A_2 B_j)\Pr(B_j)$...	$\Pr(A_2 B_b)\Pr(B_b)$	$\Pr(A_2)$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
A_i	$\Pr(A_i B_1)\Pr(B_1)$...	$\Pr(A_i B_j)\Pr(B_j)$...	$\Pr(A_i B_b)\Pr(B_b)$	$\Pr(A_i)$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
A_a	$\Pr(A_a B_1)\Pr(B_1)$...	$\Pr(A_a B_j)\Pr(B_j)$...	$\Pr(A_a B_b)\Pr(B_b)$	$\Pr(A_a)$
$\Pr(B)$	$\Pr(B_1)$...	$\Pr(B_j)$...	$\Pr(B_b)$	

Note that $\Pr(A_i)$ can be given as the row sum:

$$\Pr(A_i) = \sum_{j=1}^b \Pr(A_i|B_j)\Pr(B_j)$$

The sum of the joint probabilities = 1.

21

Posterior Probability

Table of the Posterior probability $\Pr(B_j|A_i)$

	B_1	...	B_j	...	B_b	sum
A_1	$\Pr(A_1 B_1)\Pr(B_1)/\Pr(A_1)$...	$\Pr(A_1 B_j)\Pr(B_j)/\Pr(A_1)$...	$\Pr(A_1 B_b)\Pr(B_b)/\Pr(A_1)$	1
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
A_i	$\Pr(A_i B_1)\Pr(B_1)/\Pr(A_i)$...	$\Pr(A_i B_j)\Pr(B_j)/\Pr(A_i)$...	$\Pr(A_i B_b)\Pr(B_b)/\Pr(A_i)$	1
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
A_a	$\Pr(A_a B_1)\Pr(B_1)/\Pr(A_a)$...	$\Pr(A_a B_j)\Pr(B_j)/\Pr(A_a)$...	$\Pr(A_a B_b)\Pr(B_b)/\Pr(A_a)$	1
$\Pr(B)$	$\Pr(B_1)$...	$\Pr(B_j)$...	$\Pr(B_b)$	

Normalize the Joint probability by dividing its row sum: $\Pr(A_i)$.

$$\sum_{j=1}^b \Pr(B_j|A_i) = 1$$

22

ST or SW (revisited)

- We know that:
 - 90% of the ST fans likes Tea,
 - 30% of the SW fans likes Tea.
 And
 - the ratio of ST fans vs SW fans = 1:9.
- What is the probability that a person is a ST fan if we find out that he/she likes Tea?
 - A1=Tea, A2=not Tea =coffee, B1=SW, B2=ST

23

ST or SW (CPT)

- Conditional Probability Table

Conditional Probability $P(A|B)$

	sw	st
	$P(A B1)$	$P(A B2)$
tea	0.3	0.9
coffee	0.7	0.1
sum	1	1

Prior Probability of B $P(B)$

	sw	st
	B1	B2
	0.9	0.1

24

ST or SW (joint probability table)

- Joint Probability Table

joint prob = Model prob x prior

		Joint Probability P(A,B)		Marginal Probability P(A)
		sw	st	
tea	A1	0.27	0.09	0.36
	coffe	A2	0.63	0.01
P(B)		0.9	0.1	1

- Posterior Probability as the row conditional

Posterior Probability P(B|A)

		sw	st	sum
		B1	B2	
tea	A1	0.750	0.250	1
coffe	A2	0.984	0.016	1

25

Popular Examples of Bayes Theorem

- Simple Problems
 - Smoker Testing, Coin Toss
- Three Prisoners Problem and its variations
 - normal and pathological CPT
- Monty Hall Problem
 - pathological CPT

26

Smoker Test Problem

- A blood test correctly detects smoker with probability 90% but gives a positive result for the non-smokers with probability 30%.

Assume that 50% of the population smokes.

What is the probability that a person tested is a smoker given the positive result?

- A1=positive result, A2=negative result
B1=smoker, B2=non-smoker

27

Solution of Smoker Test Problem

Smoking

CPT of A given B and Marginal B
PROB1

	B1:Smoke	B2:NonSmoke
A1:Positive	.900	.300
A2:Negative	.100	.700
Bm	.500	.500

Joint Probability of A and B and Marginal A
PROB2

	B1:Smoke	B2:NonSmoke	Am
A1:Positive	.450	.150	.600
A2:Negative	.050	.350	.400
Bm	.500	.500	1.000

Posterior Probability of B given A
PROB3

	B1:Smoke	B2:NonSmoke	sum
A1:Positive	.750	.250	1
A2:Negative	.125	.875	1
Bm	.500	.500	

28

Coin Toss Problem

- A coin was tossed and we observed a Head. Was the coin fair?
- A1=Head, A2=Tail
B1=coin is Fair, B2=coin is not Fair
- CPT:
Pr(Head | Fair) = 1/2
Pr(Head | unFair) = 2/3
- Marginal Prob
Pr(FairCoin)=1/2

29

Solution for Coin Toss Problem

CPT of A given B and Marginal of B	Joint Probability of A and B and marginal of A			Posterior Probability			
	B1:1/2	B2:2/3	Am	B1:1/2	B2:2/3	sum	
A1:Head	.50000	.66667	A1:Head .25000 .33333 .58333	A1:Head	.42857	.57143	1
A2:Tail	.50000	.33333	A2:Tail .25000 .16667 .41667	A2:Tail	.60000	.40000	1
Bm	.50000	.50000	Bm .50000 .50000 1.0000	Bm	.50000	.50000	.

30

Three Prisoners Problem (normal CPT)

- There are three prisoners, A, B, and C, one of whom will be released and the other two executed.

Prisoner A overhears that the warden say that prisoner B will be executed.

Assuming the warden's truthfulness, there are now only two possibilities for who will be executed: A or C.

Does prisoner A gain any information as to his own fate, that is, does he change his estimate of the chances he will be released?

31

Solution of Three Prisoners Problem (normal CPT)

- Let
A1=Warden says that A will be executed.
A2=Warden says that B will be executed.
A3=Warden says that C will be executed.

B1=A will be released.
B2=B will be released.
B3=C will be released.

- Let
Pr(B1)=Pr(B2)=Pr(B3)= 1/3

32

Solution of the 3 prisoners (normal CPT)

Conditional Probability $P(A|B)$

	A free	B free	C free
	$P(A B1)$	$P(A B2)$	$P(A B3)$
A dies	0	0.5	0.5
B dies	0.5	0	0.5
C dies	0.5	0.5	0
sum	1	1	1

Prior Probability $P(B)$

B1	B2	B3
0.333	0.333	0.333

Joint Probability $P(A,B)$

	A free	B free	C free	
	B1	B2	B3	$P(A)$
A dies	0.0000	0.1667	0.1667	0.3333
B dies	0.1667	0.0000	0.1667	0.3333
C dies	0.1667	0.1667	0.0000	0.3333
sum	0.3333	0.3333	0.3333	1.0000

Posterior Probability $P(B|A)$

	A free	B free	C free	
	B1	B2	B3	sum
A dies	0.000	0.500	0.500	1.000
B dies	0.500	0.000	0.500	1.000
C dies	0.500	0.500	0.000	1.000

33

Three Prisoners Problem

- There are three prisoners, A, B, and C, one of whom will be released and the other two executed.

Prisoner A asks the warden to tell him the name of one of the others (B or C) who will be executed. As the question is not directly about A's fate, the warden obliges and answers that B will be executed.

Assuming the warden's truthfulness, there are now only two possibilities for who will be executed: A or C.

Does prisoner A gain any information as to his own fate, that is, does he change his estimate of the chances he will be released?

34

Solution of the 3 prisoners 1

Conditional Probability $P(A|B)$

	A free	B free	C free
	$P(A B1)$	$P(A B2)$	$P(A B3)$
A1: A dies	0	0	0
A2: B dies	0.5	0	1
A3: C dies	0.5	1	0
sum	1	1	1

Prior Probability $P(B)$

B1	B2	B3
0.333	0.333	0.333

Joint Probability $P(A,B)$

	A free	B free	C free	
	B1	B2	B3	$P(A)$
A1: A dies	0.0000	0.0000	0.0000	0.0000
A2: B dies	0.1667	0.0000	0.3333	0.5000
A3: C dies	0.1667	0.3333	0.0000	0.5000
sum	0.3333	0.3333	0.3333	1.0000

Posterior Probability $P(B|A)$

	A free	B free	C free	
	B1	B2	B3	sum
A1: A dies	0.000	0.000	0.000	0.000
A2: B dies	0.333	0.000	0.667	1.000
A3: C dies	0.333	0.667	0.000	1.000

35

Solution of the 3 prisoners 2

Conditional Probability $P(A|B)$

	A free	B free	C free
	$P(A B1)$	$P(A B2)$	$P(A B3)$
A1: A dies	0	0	0
A2: B dies	0.5	0	1
A3: C dies	0.5	1	0
sum	1	1	1

Prior Probability $P(B)$

B1	B2	B3
0.333	0.500	0.167

Joint Probability $P(A,B)$

	A free	B free	C free	
	B1	B2	B3	$P(A)$
A1: A dies	0.0000	0.0000	0.0000	0.0000
A2: B dies	0.1667	0.0000	0.1667	0.3333
A3: C dies	0.1667	0.5000	0.0000	0.6667
sum	0.3333	0.5000	0.1667	1.0000

Posterior Probability $P(B|A)$

	A free	B free	C free	
	B1	B2	B3	sum
A1: A dies	0.000	0.000	0.000	0.000
A2: B dies	0.500	0.000	0.500	1.000
A3: C dies	0.250	0.750	0.000	1.000

36

Solution of the 3 prisoners 3

Conditional Probability P(A|B)

	A free	B free	C free
	P(A B1)	P(A B2)	P(A B3)
A1: A dies	0	0	0
A2: B dies	0.5	0	1
A3: C dies	0.5	1	0
sum	1	1	1

Joint Probability P(A,B)

	A free	B free	C free	P(A)
	B1	B2	B3	P(A)
A1: A dies	0.0000	0.0000	0.0000	0.0000
A2: B dies	0.1667	0.0000	0.5000	0.6667
A3: C dies	0.1667	0.1667	0.0000	0.3333
sum	0.3333	0.1667	0.5000	1.0000

Marginal Probability P(A)

Prior Probability P(B)

B1	B2	B3
0.333	0.167	0.500

Posterior Probability P(B|A)

	A free	B free	C free	sum
	B1	B2	B3	sum
A1: A dies	0.000	0.000	0.000	0.000
A2: B dies	0.250	0.000	0.750	1.000
A3: C dies	0.500	0.500	0.000	1.000

37

Monty Hall Problem

- Suppose you're on a game show, and you're given the choice of 3 doors:
Behind one door is a car;
behind the other 2 doors, goats.
You will get whatever behind the door you choose.

You pick a door, say "A", and the host, who knows what's behind the doors, opens another door, say "B", which has a goat. He then says to you, "Do you want to pick door "C" ?"

Is it to your advantage to switch your choice?
(Let's Make a Deal: 1963-1976 on US network TV)

38

Solution of Monty Hall Problem

- Let

A1=Host opens door A.
A2=Host opens door B.
A3=Host opens door C.

B1=Car is behind door A and you pick door A.
B2=Car is behind door B and you pick door A.
B3=Car is behind door C and you pick door A.

- Let

$$\Pr(B1)=\Pr(B2)=\Pr(B3) = 1/3$$

39

Solution

Conditional Probability P(A|B)

	Aa	Ba	Ca
	P(A B1)	P(A B2)	P(A B3)
a	0	0	0
b	0.5	0	1
c	0.5	1	0
sum	1	1	1

Joint Probability P(A,B) Marginal Probability P(A)

	Aa	Ba	Ca	P(A)
	B1	B2	B3	P(A)
a	0.000	0.000	0.000	0.0000
b	0.167	0.000	0.333	0.5000
c	0.167	0.333	0.000	0.5000
sum	0.333	0.333	0.333	1.0000

Prior Probability of B (P(B))

B1	B2	B3
0.333	0.333	0.333

Posterior Probability P(B|A)

	Aa	Ba	Ca	sum
	B1	B2	B3	sum
a	0.000	0.000	0.000	0.000
b	0.333	0.000	0.667	1.000
c	0.333	0.667	0.000	1.000

Aa = Car is behind door A and you pick door a.
Ba = Car is behind door B and you pick door a.
Ca = Car is behind door C and you pick door a.

a = Host opens door A
b = Host opens door B
c = Host opens door C

40

Test Scores

- A student can be classified as L, M, or H according to his/her ability.

The CPT of a five item test score given ability level is known.

Also, Pr(L), Pr(M) and Pr(H) are known.

- Given the test score (0-5) how do we classify the student?

41

Solution

Let A_i be "test score = $i - 1$ " and

$B_1=L, B_2=M, B_3=H$.

Also, we know that $\Pr(B_1) = \Pr(B_3) = 0.25, \Pr(B_2) = .5$.

CPT of A given B				Joint Probability of A and B				Posterior Probability					
PROB1	B1:L	B2:M	B3:H	PROB2	B1:L	B2:M	B3:H	Am	PROB3	B1:L	B2:M	B3:H	sum
A1:0	.10	.10	.00	A1:0	.025	.050	.000	.075	A1:0	.33333	.66667	.00000	1
A2:1	.30	.10	.00	A2:1	.075	.050	.000	.125	A2:1	.60000	.40000	.00000	1
A3:2	.40	.20	.10	A3:2	.100	.100	.025	.225	A3:2	.44444	.44444	.11111	1
A4:3	.20	.30	.20	A4:3	.050	.150	.050	.250	A4:3	.20000	.60000	.20000	1
A5:4	.00	.20	.40	A5:4	.000	.100	.100	.200	A5:4	.00000	.50000	.50000	1
A6:5	.00	.10	.30	A6:5	.000	.050	.075	.125	A6:5	.00000	.40000	.60000	1
Bm	.25	.50	.25	Bm	.250	.500	.250	1.00	Bm	.25000	.50000	.25000	

For example, if a person's score is 3, then his/her probability of belonging to class M = 0.6. $\Pr(M | 3) = 0.6$

Multiple Independent Observations

$$\Pr(A_{i1}, A_{i2} | B_j) = \Pr(A_{i1} | B_j) \Pr(A_{i2} | B_j)$$

CPT or Model Probability					
	B_1	...	B_j	...	B_b
A_1, A_1	$\Pr(A_1 B_1)\Pr(A_1 B_1)$...	$\Pr(A_1 B_j)\Pr(A_1 B_j)$...	$\Pr(A_1 B_b)\Pr(A_1 B_b)$
A_1, A_2	$\Pr(A_1 B_1)\Pr(A_2 B_1)$...	$\Pr(A_1 B_j)\Pr(A_2 B_j)$...	$\Pr(A_1 B_b)\Pr(A_2 B_b)$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
A_{i1}, A_{i2}	$\Pr(A_{i1} B_1)\Pr(A_{i2} B_1)$...	$\Pr(A_{i1} B_j)\Pr(A_{i2} B_j)$...	$\Pr(A_{i1} B_b)\Pr(A_{i2} B_b)$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
A_a, A_a	$\Pr(A_a B_1)\Pr(A_a B_1)$...	$\Pr(A_a B_j)\Pr(A_a B_j)$...	$\Pr(A_a B_b)\Pr(A_a B_b)$
$\Pr(B)$	$\Pr(B_1)$...	$\Pr(B_j)$...	$\Pr(B_b)$

If $A_i, i = 1, 2, \dots, a$ are mutually exclusive and exhaustive

$$\sum_{i1=1}^a \sum_{i2=1}^a \Pr(A_{i1}, A_{i2} | B_j) = 1$$

Joint Probability

$$\Pr(A_{i1}, A_{i2}, B_j) = \Pr(A_{i1}, A_{i2} | B_j) \times \Pr(B_j)$$

Posterior Probability

$$\Pr(B_j | A_{i1}, A_{i2}) = \frac{1}{\sum_{j=1}^b \Pr(A_{i1}, A_{i2}, B_j)} \Pr(A_{i1}, A_{i2}, B_j)$$

43

Coin Tossed Twice

CPT of A given B and Marginal of B			Joint Probability of A and B and marginal of A			Posterior Probability				
B1:1/2	B2:2/3		B1:1/2	B2:2/3	Am	B1:1/2	B2:2/3	sum		
A1:HH	.25000	.44444	A1:HH	.12500	.22222	.34722	A1:HH	.36000	.64000	1.
A2:HT	.25000	.22222	A2:HT	.12500	.11111	.23611	A2:HT	.52941	.47059	1.
A3:TH	.25000	.22222	A3:TH	.12500	.11111	.23611	A3:TH	.52941	.47059	1.
A4:TT	.25000	.11111	A4:TT	.12500	.05556	.18056	A4:TT	.69231	.30769	1.
Bm	.50000	.50000	Bm	.50000	.50000	1.0000	Bm	.50000	.50000	

Note that the posterior odds of not Fair to Fair, given Head, is larger with this example.

$$\frac{\Pr(\text{notFair} | \text{Head})}{\Pr(\text{Fair} | \text{Head})} = \frac{.571}{.429} = 1.333 < 1.778 = \frac{.640}{.360} = \frac{\Pr(\text{notFair} | \text{Head}, \text{Head})}{\Pr(\text{Fair} | \text{Head}, \text{Head})}$$

Note also that, $\Pr(\text{Fair} | \text{Head}, \text{Tail}) = \Pr(\text{Fair} | \text{Tail}, \text{Head})$.

Number of Heads is sufficient for the inference.

4

Two Test Scores

Test Score (two scores)				Joint Probability of A and B and the Marginal					Posterior Probability			
Conditional Probability of A given B (Model)				PROB2 B1:L B2:M B3:H Am					PROB3 B1:L B2:M B3:H			
PROB1	B1:L	B2:M	B3:H	PROB2	B1:L	B2:M	B3:H	Am	PROB3	B1:L	B2:M	B3:H
A1:00	.01000	.01000	.00000	A1:00	.00250	.00500	.00000	.00750	A1:00	.33333	.66667	.00000
A2:01	.03000	.01000	.00000	A2:01	.00750	.00500	.00000	.01250	A2:01	.60000	.40000	.00000
A3:02	.04000	.02000	.00000	A3:02	.01000	.01000	.00000	.02000	A3:02	.50000	.50000	.00000
A4:03	.02000	.03000	.00000	A4:03	.00500	.01500	.00000	.02000	A4:03	.25000	.75000	.00000
A5:04	.00000	.02000	.00000	A5:04	.00000	.01000	.00000	.01000	A5:04	.00000	1.0000	.00000
A6:05	.00000	.01000	.00000	A6:05	.00000	.00500	.00000	.00500	A6:05	.00000	1.0000	.00000
A7:10	.03000	.01000	.00000	A7:10	.00750	.00500	.00000	.01250	A7:10	.60000	.40000	.00000
A8:11	.09000	.01000	.00000	A8:11	.02250	.00500	.00000	.02750	A8:11	.81818	.18182	.00000
A9:12	.12000	.02000	.00000	A9:12	.03000	.01000	.00000	.04000	A9:12	.75000	.25000	.00000
A10:13	.06000	.03000	.00000	A10:13	.01500	.01500	.00000	.03000	A10:13	.50000	.50000	.00000
A11:14	.00000	.02000	.00000	A11:14	.00000	.01000	.00000	.01000	A11:14	.00000	1.0000	.00000
A12:15	.00000	.01000	.00000	A12:15	.00000	.00500	.00000	.00500	A12:15	.00000	1.0000	.00000
A13:20	.04000	.02000	.00000	A13:20	.01000	.01000	.00000	.02000	A13:20	.50000	.50000	.00000
A14:21	.12000	.02000	.00000	A14:21	.03000	.01000	.00000	.04000	A14:21	.75000	.25000	.00000
A15:22	.16000	.04000	.01000	A15:22	.04000	.02000	.00250	.06250	A15:22	.64000	.32000	.04000
A16:23	.08000	.06000	.02000	A16:23	.02000	.03000	.00500	.05500	A16:23	.36364	.54545	.09091
A17:24	.00000	.04000	.04000	A17:24	.00000	.02000	.01000	.03000	A17:24	.00000	.66667	.33333
A18:25	.00000	.02000	.03000	A18:25	.00000	.01000	.00750	.01750	A18:25	.00000	.57143	.42857
A19:30	.02000	.03000	.00000	A19:30	.00500	.01500	.00000	.02000	A19:30	.25000	.75000	.00000
A20:31	.06000	.03000	.00000	A20:31	.01500	.01500	.00000	.03000	A20:31	.50000	.50000	.00000
A21:32	.08000	.06000	.02000	A21:32	.02000	.03000	.00500	.05500	A21:32	.36364	.54545	.09091
A22:33	.04000	.09000	.04000	A22:33	.01000	.04500	.01000	.06500	A22:33	.15385	.69231	.15385
A23:34	.00000	.08000	.08000	A23:34	.00000	.03000	.02000	.05000	A23:34	.00000	.60000	.40000
A24:35	.00000	.03000	.06000	A24:35	.00000	.01500	.01500	.03000	A24:35	.00000	.50000	.50000
A25:40	.00000	.02000	.00000	A25:40	.00000	.01000	.00000	.01000	A25:40	.00000	1.0000	.00000
A26:41	.00000	.02000	.00000	A26:41	.00000	.01000	.00000	.01000	A26:41	.00000	1.0000	.00000
A27:42	.00000	.04000	.04000	A27:42	.00000	.02000	.01000	.03000	A27:42	.00000	.66667	.33333
A28:43	.00000	.06000	.08000	A28:43	.00000	.03000	.02000	.05000	A28:43	.00000	.60000	.40000
A29:44	.00000	.04000	.16000	A29:44	.00000	.02000	.04000	.06000	A29:44	.00000	.33333	.66667
A30:45	.00000	.02000	.12000	A30:45	.00000	.01000	.03000	.04000	A30:45	.00000	.25000	.75000
A31:50	.00000	.01000	.00000	A31:50	.00000	.00500	.00000	.00500	A31:50	.00000	1.0000	.00000
A32:51	.00000	.01000	.00000	A32:51	.00000	.00500	.00000	.00500	A32:51	.00000	1.0000	.00000
A33:52	.00000	.02000	.03000	A33:52	.00000	.01000	.00750	.01750	A33:52	.00000	.57143	.42857
A34:53	.00000	.03000	.06000	A34:53	.00000	.01500	.01500	.03000	A34:53	.00000	.50000	.50000
A35:54	.00000	.02000	.12000	A35:54	.00000	.01000	.03000	.04000	A35:54	.00000	.25000	.75000
A36:55	.00000	.01000	.09000	A36:55	.00000	.00500	.02250	.02750	A36:55	.00000	.18182	.81818
Bm	.25000	.50000	.25000	Bm	.25000	.50000	.25000	1.0000	Bm	.25000	.50000	.25000

Two Test Scores

Note that, with this example (CPT) the sum of two test scores is not sufficient. For example the posterior probability of B given sum = 6 are shown below.

Conditional Probability of A given B (Model)				Joint Probability of A and B and the Marginal					Posterior Probability			
PROB1	B1:L	B2:M	B3:H	PROB2	B1:L	B2:M	B3:H	Am	PROB3	B1:L	B2:M	B3:H
A12:15	.00000	.01000	.00000	A12:15	.00000	.00500	.00000	.00500	A12:15	.00000	1.0000	.00000
A32:51	.00000	.01000	.00000	A32:51	.00000	.00500	.00000	.00500	A32:51	.00000	1.0000	.00000
A17:24	.00000	.04000	.04000	A17:24	.00000	.02000	.01000	.03000	A17:24	.00000	.66667	.33333
A27:42	.00000	.04000	.04000	A27:42	.00000	.02000	.01000	.03000	A27:42	.00000	.66667	.33333
A22:33	.04000	.09000	.04000	A22:33	.01000	.04500	.01000	.06500	A22:33	.15385	.69231	.15385

If CPT is proportional to the PDF of the Normal distribution, sum is the sufficient statistics.